Exam 4 Friday 11.1–11.6

Final Exam Tuesday 7-31

Omitted: 7.2, 7.5, 8.6, 10.5–10.7

Priorities for study:

1. 11.7–11.10
2. Chapter 8
3. Chapter 6
4. 11.1–11.6 again

4 notecards <u>or</u> 1 $8\frac{1}{2} \times 11$ sheet (both sides)

11.2 29) $\displaystyle\sum_{n=0}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} (e^{-2})^n$ geometric series (i.e. of the form $\displaystyle\sum_{n=0}^{\infty} ar^n$, $a=1$ $r=e^{-2}$)

$$\boxed{\frac{1}{e^{2n}} \to 0 \text{ as } n \to \infty. \text{ but this is not enough}}$$

$|r| = |e^{-2}| = e^{-2} < 1$

so <u>converges</u>.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad (\text{if } |r|<1)$$

$$\sum_{n=0}^{\infty} e^{-2n} = \frac{1}{1-e^{-2}} = \frac{e^2}{e^2-1} \quad //$$

11.3 9) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$

$$\boxed{\frac{\ln(n)}{n} \to 0 \text{ as } n \to \infty}$$

<u>Integral test:</u> $f(x) = \dfrac{\ln(x)}{x}$ decreasing.

$\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ /div with $\displaystyle\int_2^{\infty} \frac{\ln(x)}{x}\,dx$

$$\int_2^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{\ln(x)}{x}\,dx$$

$\displaystyle\int_2^{b} \frac{\ln x}{x}\,dx$ $\left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \\ x=2 \quad u = \ln 2 \\ x = b \quad u = \ln b \end{array}\right.$ $\displaystyle = \int_{\ln 2}^{\ln b} u\,du$

$$= \frac{1}{2}u^2 \Big|_{\ln 2}^{\ln b}$$

$$= \frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln 2)^2$$

$$= \lim_{b\to\infty} \frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln 2)^2 = \infty$$

so integral diverges.

Direct comparison Compare with $\frac{1}{n}$.

$$\frac{1}{n} \le \frac{\ln(n)}{n} \quad \text{if } n \ge 3$$

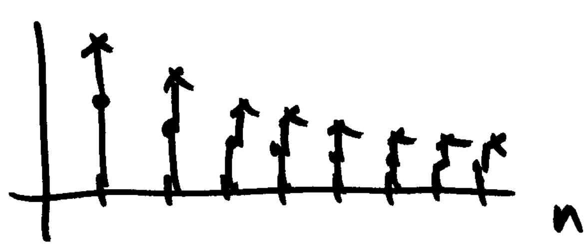
$$\therefore \underbrace{\sum_{n=3}^{\infty} \frac{1}{n}}_{\substack{\text{divergent} \\ p\text{-series} \\ p=1}} \le \sum_{n=3}^{\infty} \frac{\ln n}{n}$$

$\therefore$ this series diverges.

Limit comparison: Compare with $\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{\ln n}{n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

so $\frac{1}{n} \to 0$ faster than $\frac{\ln n}{n} \to 0$



$\bullet$ – $\frac{1}{n}$

$\times$ – $\frac{\ln n}{n}$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges

(p-series $p=1$)

so does $\sum_{n=2}^{\infty} \frac{\ln n}{n}$.

11.4 11) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

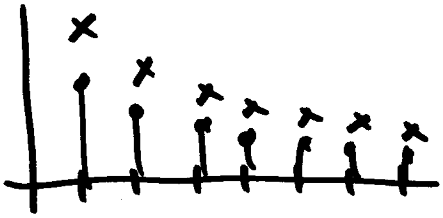
<u>Direct</u> <u>comp.</u> with $\frac{1}{n^3}$

$$\frac{1}{n^3} \le \frac{(\ln n)^2}{n^3} \quad \text{NO GOOD.}$$

$$\underbrace{\sum_{n=3}^{\infty} \frac{1}{n^3}}_{\substack{\text{finite} \\ \text{p-series} \\ p=3>1}} \le \underbrace{\sum_{n=3}^{\infty} \frac{(\ln n)^2}{n^3}}_{\substack{\text{gives me} \\ \text{no information} \\ \text{about this}}}$$

<u>Limit</u> <u>comp.</u> compare with $\frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} (\ln n)^2 = \infty \quad \underline{\text{NO GOOD}}$$



Gives no information.

$\bullet$ - $\frac{1}{n^3}$

x - $\frac{(\ln n)^2}{n^3}$

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \frac{(\ln n)^2}{n}$$

$$\to 0$$

<u>Dir</u>ect <u>comp</u> with $\frac{1}{n^2}$

$$\frac{(\ln n)^2}{n^3} = \frac{1}{n^2} \cdot \frac{(\ln n)^2}{n} \leq \frac{1}{n^2} \quad \text{eventually.}$$

$$\therefore \underbrace{\sum_{n=N}^{\infty} \frac{(\ln n)^2}{n^3}}_{\text{converges.}} \leq \sum_{n=N}^{\infty} \frac{1}{n^2} < \infty$$

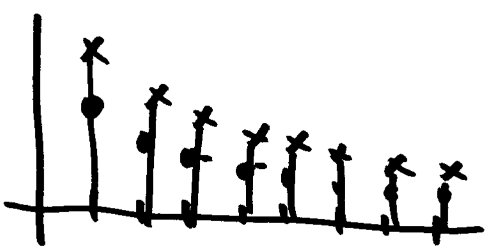
(p-series, $p = 2 > 1$)

<u>Limit</u> c<u>omp</u> with $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

so does $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$



$\bullet$ - $\frac{(\ln n)^2}{n^3}$

$\times$ - $\frac{1}{n^2}$

11.4 19) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ looks like $\frac{1}{n\sqrt{n^2}} = \frac{1}{n^2}$

Direct comp. with $\frac{1}{n^2}$

Want: $\frac{1}{n\sqrt{n^2-1}} \le \frac{1}{n^2}$ NOT TRUE

Limit comp is better (compare with $\frac{1}{n^2}$)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n\sqrt{n^2-1}}} = \lim_{n\to\infty} \frac{\cancel{n}\sqrt{n^2-1}}{n^{\cancel{2}}} = \lim_{n\to\infty} \sqrt{\frac{n^2-1}{n^2}} = 1$$

$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges with $\sum_{n=2}^{\infty} \frac{1}{n^2}$

25) $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$

$$\boxed{\lim_{n\to\infty} \sin(\tfrac{1}{n}) = \sin(0) = 0.}$$

Fact: Recall

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

So

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$$

By limit comp (with $\frac{1}{n}$), series diverges with $\sum_{n=1}^{\infty} \frac{1}{n}$

11.5 7) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{(1.25)^n}$

behaves like $\sum_{n=1}^{\infty} \frac{1}{(1.25)^n} = \sum_{n=1}^{\infty} \left(\frac{1}{1.25}\right)^n$

$\frac{1}{1.25} < 1$ so convergent geom. series.

Direct comp with $\frac{3}{(1.25)^n}$

$$\frac{2+(-1)^n}{(1.25)^n} \leq \frac{3}{(1.25)^n}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{2+(-1)^n}{(1.25)^n}}_{\text{converges also.}} \leq \sum_{n=1}^{\infty} \frac{3}{(1.25)^n} < \infty$$

because convergent geom. series.

13) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$

Could say $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n^2} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\substack{\text{infinite} \\ \text{p-series} \\ p=1}} - \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{finite} \\ \text{p-series} \\ p=2>1}} = \infty$

OR $\frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2}$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{n-1}{n^2}$$ ← limit comp with $\frac{1}{n}$.

$\therefore$ series diverges with $\sum_{n=1}^{\infty} \frac{1}{n}$.

15) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

21) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

Ratio test: $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{n!}$

$$= \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!} = (n+1) \cdot \frac{1}{(2n+3)(2n+2)}$$

$$\frac{(2n+1)!}{(2n+3)!} = \frac{\cancel{(2n+1)(2n)(2n-1)(2n-2)\cdots(2)(1)}}{(2n+3)(2n+2)\cancel{(2n+1)(2n)(2n-1)\cdots(2)(1)}}$$

$$= \frac{1}{(2n+3)(2n+2)}$$

$$= \frac{\cancel{n+1}}{(2n+3)(2)\cancel{(n+1)}} = \frac{1}{2(2n+3)} \xrightarrow[n\to\infty]{} 0 < 1$$

$\therefore$ series converges.

25) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,\ln(n)}{n(n+2)!}$

<u>Ratio test</u>

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\,\ln(n+1)}{(n+1)(n+3)!} \cdot \frac{n\,(n+2)!}{n!\,\ln(n)} =$$

$$= \cancel{(n+1)} \cdot \frac{n}{\cancel{n+1}} \cdot \frac{1}{n+3} \cdot \frac{\ln(n+1)}{\ln(n)}$$

$$= \underbrace{\frac{n}{n+3}}_{\to 1} \cdot \underbrace{\frac{\ln(n+1)}{\ln(n)}}_{\to 1} \longrightarrow 1 \quad \boxed{\text{Test fails}}$$

as $n \to \infty$.

$$\boxed{\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = 1}$$

$$\sum_{n=1}^{\infty} \frac{n!\,\ln(n)}{n\,(n+2)!} = \sum_{n=1}^{\infty} \frac{\ln(n)}{n(n+1)(n+2)}$$ behaves like $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

$$\boxed{\frac{n!}{(n+2)!} = \frac{\cancel{n(n-1)(n-2)\cdots(1)}}{(n+2)(n+1)\cancel{(n)(n-1)\cdots(1)}} = \frac{1}{(n+2)(n+1)}}$$

can do limit comp with $\frac{1}{n^2}$ like before.

Conclude: the series <u>converges</u>.

<u>11.6</u> 33) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/2}}$

$\boxed{\text{In fact } \cos n\pi = (-1)^n. \text{ Do we care? NOT REALLY}}$

Can look at

$$\sum_{n=1}^{\infty} \left| \frac{\cos n\pi}{n^{3/2}} \right| = \sum_{n=1}^{\infty} \frac{|\cos(n\pi)|}{n^{3/2}}$$

Direct comp with $\frac{1}{n^{3/2}}$

$$\frac{|\cos(n\pi)|}{n^{3/2}} \le \frac{1}{n^{3/2}} \;; \quad \underbrace{\sum_{n=1}^{\infty} \frac{|\cos(n\pi)|}{n^{3/2}}}_{\text{converges}} \le \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$$

p-series $p = 3/2 > 1$

So $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}}$ converges absolutely.

39) $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}-\sqrt{n}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$

$$(\sqrt{n+1}-\sqrt{n}) \cdot \frac{(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

behaves like $\sum \frac{1}{2\sqrt{n}}$ which diverges.

Limit comparison with $\frac{1}{2\sqrt{n}}$

says series diverges with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$

p-series $p=\frac{1}{2}<1$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ does <u>not</u> converge absolutely.

But $\sum_{n\geq 1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ converges by A.S.T.

because $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ decreases to zero as $n \to \infty$.

$\therefore$ series <u>converges</u> con<u>ditionally</u>.

11.9 Convergence of Taylor series

Recall: ① Given $f(x)$ we can write its Taylor

series at $x=a$ $\quad \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$

② Since this is a power series we can find its radius of convergence and its interval of convergence. So we know for which x the Taylor series converges.

③ Does the series converge to $f(x)$?

That is, can we say

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

whenever the series converges?

e.g. We know Taylor series for e^x at $x=0$ is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. But do we know that in fact $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$?

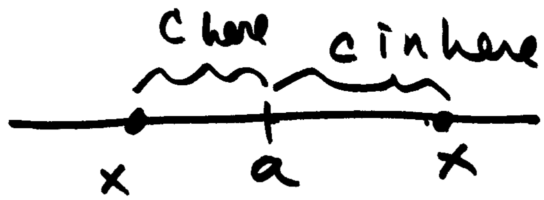
Taylor's formula. (p 796)

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

What is the remainder? That is, what is $f(x) - P_n(x) = R_n(x)$.

In fact $$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c between x and a.



e.g. Taylor series for $\sin(x)$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

$$P_7(x) = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

$$\sin(x) - P_7(x) = R_7(x) = \frac{1}{8!} \frac{d^8}{dx^8}(\sin(x))\Big|_{x=c}(x-0)^8$$

$$\frac{d^8}{dx^8}\sin(x) = \sin(x)$$

So $R_7(x) = \frac{1}{8!}\sin(c)\,x^8$ some c between x and 0.

Since $|\sin(c)| \le 1$ we know

$$|R_7(x)| \le \frac{1}{8!}(1)|x|^8 = \frac{|x|^8}{8!}$$

large if x is away from zero

small if x near zero

Notice: For $\sin(x)$

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0$$

as $n \to \infty$ for every x.

This means

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

for every x.

Similarly you can show that

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all x and

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \text{ for all } x.$$

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e.g. #2 $e^{-x/2}$

Know $e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$

$$\therefore \; e^{-x/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n$$

So just for fun: if $x=1$, then

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} = e^{-1/2}$$

#6) $\cos\left(\frac{x^{3/2}}{\sqrt{2}}\right)$

$$\cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}$$

$$\therefore \cos\left(\frac{x^{3/2}}{\sqrt{2}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{x^{3/2}}{\sqrt{2}}\right)^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{2^n} x^{3n}$$

#12) $x^2 \cos(x^2)$

$$x^2 \cos(x^2) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n}$$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+2}$$