MAPLE #3 due Tomorrow

Exam 4 Friday 11.1 – 11.6

Final Exam Tuesday July 31

1. omit Chapter 10 (maybe more)
2. 11.7 – 11.10 will be prominent

Friday class ends 12:20

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$\{a_n\}_{n=1}^{\infty}$ $\quad \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

Q: Does series converge? This means that $\lim_{n\to\infty} S_n$ exists (i.e. as a finite #) where $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$

$\{S_n\}_{n=1}^{\infty}$ = sequence of partial sums

$\{a_n\}_{n=1}^{\infty}$ = sequence of terms

For convergence we need $\lim_{n\to\infty} a_n = 0$ and they must go to zero fast enough.

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11.7 Power series

1. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

$a \longleftrightarrow$ center of series

$\{c_n\} \longleftrightarrow$ sequence of coefficients

Note: (a) series starts at $n=0$.

(b) the series defines a function of $x$.

(c) the terms $c_n(x-a)^n$ suggest geometric series behavior.

(d) $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

The partial sums are polynomials. In fact

$S_n = \sum_{k=0}^{n-1} c_k(x-a)^k$ is a poly of degree $n-1$.

e.g. $\sum_{n=0}^{\infty} x^n$ $\quad a=0$ (center of power series)

$c_n = 1$ for all $n$.

This is a geometric series, converging when $|x|<1$ to $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Idea: $\frac{1}{1-x} \approx \sum_{k=0}^{n-1} x^k = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$

when $n$ is large.

Recall: $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$

**Q:** Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ for which $x$ does it converge?

eg #2) $\sum_{n=0}^{\infty}(x+5)^n$ Apply ratio test:

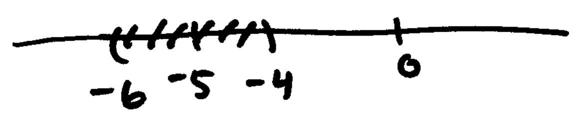
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| = |x+5|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x+5|$$

$\therefore \sum_{n=0}^{\infty}(x+5)^n$ conv. absolutely if $|x+5|<1$, i.e. if

$$-1 < x+5 < 1$$

$$\underline{-6 < x < -4}$$




**Also** $\sum_{n=0}^{\infty}(x+5)^n$ diverges when $|x+5|>1$

Test fails when $|x+5|=1$, i.e. when $x=-4$ or $x=-6$

so we check separately:

$x=-4$: $x+5=1$, $\sum_{n=0}^{\infty}(x+5)^n = \sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ diverges

$x=-6$: $x+5=-1$, $\sum_{n=0}^{\infty}(x+5)^n = \sum_{n=0}^{\infty}(-1)^n$ diverges also.

Conclusion: $\sum_{n=0}^{\infty}(x+5)^n$ converges when $x$ is in $(-6,-4)$ and diverges elsewhere.

Interval of convergence.

e.g. #8) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n}$

Power series:
center $a=-2$
$c_n=\dfrac{(-1)^n}{n}$

For which $x$ does it converge?

Ratio test: $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(x+2)^{n+1}}{n+1}}{\frac{(-1)^n(x+2)^n}{n}}\right| = \left|\frac{(x+2)^{n+1}}{n+1}\cdot\frac{n}{(x+2)^n}\right|$$

$$= \left|(x+2)\frac{n}{n+1}\right| = |x+2|\underbrace{\left(\frac{n}{n+1}\right)}_{1} \xrightarrow[\text{as } n\to\infty]{} |x+2|$$

$\therefore \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n}$ conv. abs. for $|x+2|<1$

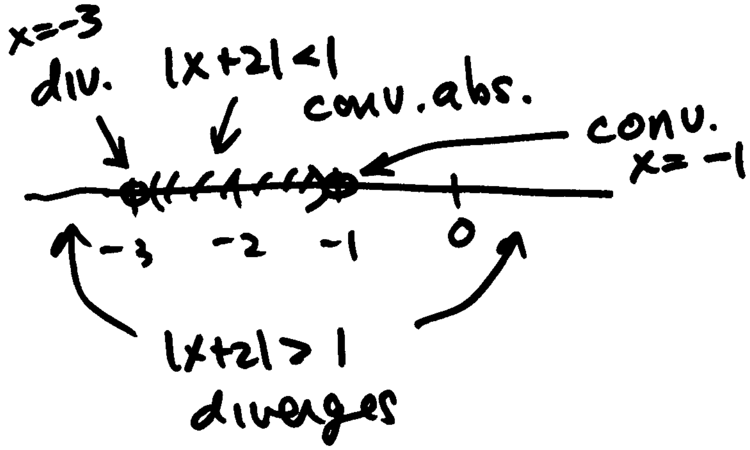
diverges for $|x+2|>1$

What about $|x+2|=1$, i.e. $x=-1$
$x=-3$

$x=-1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(1)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ <u>converges</u> by A.S.T.

$x=-3$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n}=\sum_{n=1}^{\infty}\frac{1}{n}$ <u>diverges</u>, p-series $p\le 1$.

Conclusion: $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$ converges for $x$ in $(-3, -1]$.




#12) $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ center $a = 0$

$c_n = \frac{3^n}{n!}$

$\boxed{0! = 1 \text{ by definition even though it makes no sense}}$

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^{n+1} x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}}\right| = \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right|$$

$$= \left|3 \cdot x \cdot \frac{1}{n+1}\right| \xrightarrow[n\to\infty]{} 0$$

$$\boxed{\frac{n!}{(n+1)!} = \frac{\cancel{n(n-1)(n-2)\cdots(2)(1)}}{(n+1)\cancel{(n)(n-1)(n-2)\cdots(2)(1)}} = \frac{1}{n+1}}$$

$\therefore \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ conv. absolutely for all $x$ in $(-\infty, \infty)$

#24) $\sum_{n=0}^{\infty} n!(x-4)^n$ center $a=4$

$c_n = n!$

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(x-4)^{n+1}}{n!(x-4)^n}\right| = |(n+1)(x-4)|$$

$$= (n+1)|x-4| \xrightarrow[\text{as } n\to\infty]{} \begin{cases} \infty & \text{if } x \neq 4 \\ 0 & \text{if } x = 4 \end{cases}$$

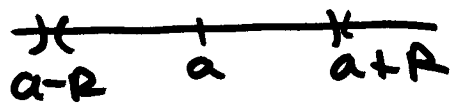
$\therefore \sum_{n=0}^{\infty} n!(x-4)^n$ converges only at $x=4$.

## Radius + Interval of Convergence

Given $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are 3 possibilities

① There is an $R>0$ such that series conv. absolutely when $|x-a|<R$

(i.e. $a-R < x < a+R$)




and diverges when $|x-a|>R$.

② Series conv. absolutely for every $x$ (say $R=\infty$).

③ Series converges only at $x=a$ (say $R=0$)

R is called the radius of convergence of power series.

Interval of convergence will be

$(a-R, a+R)$ depending on whether
$[a-R, a+R)$ series converges at
$(a-R, a+R]$ $x-a = \pm R$.
$[a-R, a+R]$ (i.e. at $|x-a| = R$)

## 11.8 Taylor and Maclaurin series

Idea: We have seen that some functions can be written as power series.

e.g. $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

e.g. #34 p788

$$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = 1 + \frac{(x+1)^2}{9} + \frac{(x+1)^4}{9^2} + \frac{(x+1)^6}{9^3} + \cdots$$

This is a power series with center $a=-1$ and coefficients

$$C_m = \begin{cases} 0 & m \text{ odd} \\ \frac{1}{9^{\frac{m}{2}}} & m \text{ even} \end{cases}$$

Then $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{m=0}^{\infty} c_m (x+1)^m$

$$= \underset{m=0}{1} + \underset{m=1}{0} + \underset{m=2}{\frac{(x+1)^2}{9}} + 0 + \frac{(x+1)^4}{9^2} + \cdots$$

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Anyway...

$$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left[\frac{(x+1)^2}{9}\right]^n = \frac{1}{1-\frac{(x+1)^2}{9}}$$

$r$

$$= \frac{9}{9-(x+1)^2} = \frac{9}{9-x^2-2x-1} = \frac{-9}{x^2+2x-8}$$

$$= \frac{-9}{(x-2)(x+4)}$$

Aside

$\displaystyle\sum_{n=0}^{\infty} ar^n$

$S_n = \underbrace{a + ar + ar^2 + \cdots + ar^{n-1}}_{n \text{ terms}}$

$= a\dfrac{1-r^n}{1-r}$

$\displaystyle\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} S_n$ $\to 0$

$= \displaystyle\lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r}$

Converges on:

$$\left|\frac{(x+1)^2}{9}\right| < 1$$

$$\frac{(x+1)^2}{9} < 1$$

$$(x+1)^2 < 9$$

$$|x+1| < 3$$

This is interval $(-4, 2)$

Question: Can any function be written as a power series? Like e.g. $e^x$? $\sin(x)$? $\ln(x)$? etc...

Answer:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

the center a is fixed. How do the $a_n$ depend on f?

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots$$

$\underline{x=a}$

$f(a) = a_0 \qquad a_0 = f(a)$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots$$

$\underline{x=a}$

$f'(a) = a_1 \qquad a_1 = f'(a)$

$$f''(x) = 2a_2 + 2\cdot 3a_3(x-a) + 3\cdot 4a_4(x-a)^2 + \cdots$$

$\underline{x=a}$

$f''(a) = 2a_2 \qquad a_2 = \frac{1}{2} f''(a)$

$$f'''(x) = 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4(x-a) + \cdots$$

$\underline{x=a}$

$f'''(a) = 2\cdot 3a_3 \qquad a_3 = \frac{1}{2\cdot 3} f'''(a)$

etc...

End up with:

$$a_n = \frac{1}{n!} f^{(n)}(a)$$

<u>Taylor series</u>

The Taylor series of $f(x)$ <u>at $x=a$</u> is the power series
$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{2\cdot 3} f'''(a)(x-a)^3 + \cdots$$

If <u>$a=0$</u> it is called the <u>Maclaurin</u> series

<u>Taylor polynomial</u>

The Taylor polynomial of order $n$ (or degree $n$) at $x=a$ is

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

e.g. Find Taylor polynomial of $e^x$ of order 0, 1, 2, 3 at $x=0$.

$$f(x) = e^x \qquad f(0) = 1$$
$$f'(x) = e^x \qquad f'(0) = 1$$
$$f''(x) = e^x \qquad f''(0) = 1$$
$$f'''(x) = e^x \qquad f'''(0) = 1$$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)x = 1 + x$$

$$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + x + \frac{1}{2}x^2$$

$$P_3(x) = P_2(x) + \frac{1}{6}f'''(0)x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

Taylor series:

$$1 + x + \frac{1}{2}x^2 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 4}x^4 + \cdots + \frac{1}{n!}x^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

e.g. Same for $\sin(x)$ at $x=0$.

$$f(x)=\sin(x) \qquad f(0)=0$$
$$f'(x)=\cos(x) \qquad f'(0)=1$$
$$f''(x)=-\sin(x) \qquad f''(0)=0$$
$$f'''(x)=-\cos(x) \qquad f'''(0)=-1$$
$$f^{(4)}(x)=\sin(x) \qquad f^{(4)}(0)=0$$

$$P_0(x)=f(0)=0$$
$$P_1(x)=f(0)+f'(0)x = x$$
$$P_2(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2 = x$$
$$P_3(x)=P_2(x)+\frac{1}{6}f'''(0)x^3 = x-\frac{1}{6}x^3$$

$$x-\frac{1}{6}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\cdots$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$ Taylor series for $\sin(x)$ at $x=0$.

e.g. Same for $\cos(x)$ at $x=0$.

Since $\frac{d}{dx}(\sin(x)) = \cos(x)$, then

$$\frac{d}{dx}\Big(\underbrace{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots}_{}\Big)$$

Taylor series for $\sin(x)$

$$= 1 - \frac{3}{3!}x^2 + \frac{5}{5!}x^4 - \frac{7}{7!}x^6 + \frac{9}{9!}x^8 - \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$$

Taylor series for $\cos(x)$ at $x=0$.

e.g. #18 $\sinh(x) = \dfrac{e^x - e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

$e^x$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ $\qquad$ $e^{-x}$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n x^n$$

$$\frac{1}{2}e^x - \frac{1}{2}e^{-x} = \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(x^n - (-1)^n x^n\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}x^n \underbrace{\left(1-(-1)^n\right)}_{0 \text{ if } n \text{ even, } 2 \text{ if } n \text{ odd}}$$

$$= \cancel{\frac{1}{x}} \cdot \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{2m+1} \cdot \cancel{x}$$

$$= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{2m+1}$$

compare with Taylor series for $\sin(x)$

which is $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$