MAPLE #3 DUE THURSDAY

EXAM #4 FRIDAY 11.1–11.6

$\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$ for all n (<u>sequence of terms</u> positive)

This means that $\{S_n\}_{n=1}^{\infty}$ is a <u>non-decreasing</u> sequence where $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$. $\{S_n\}$ is the <u>sequence of partial sums.</u> $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$.

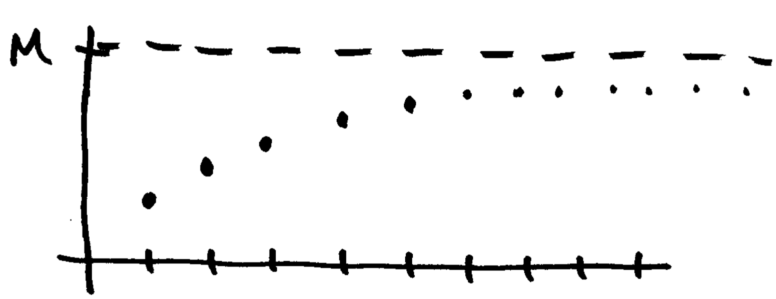
For a series to converge we need:

1. $\lim_{n\to\infty} a_n = 0$ <u>and</u>
2. $a_n \to 0$ fast enough.

<u>e.g.</u> $\sum_{n=1}^{\infty} \frac{1}{n}$ $\qquad \lim_{n\to\infty} \frac{1}{n} = 0$

but $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

<u>Fact</u>: If a non-decreasing sequence has an upper bound, then it converges. i.e. $S_n \leq M$ for all n



we have looked at integral test:

$$\sum_{k=1}^{n} a_k \le \int_1^n f(x)\,dx + a_1 \le \int_1^\infty f(x)\,dx + 1$$

where $a_n = f(n)$, $f(x)$ is decreasing

11.5 Ratio and Root tests.

1. Ratio test.

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

① Exponential (2^n) beats polynomial (n^3) so $\displaystyle\lim_{n\to\infty} \frac{n^3}{2^n} = 0$.

② Q: Do terms go to zero fast enough? YES

e.g. integral test: $\displaystyle\int_1^\infty x^3 2^{-x}\,dx$ this is doable (by parts) so this works.

③ Another idea: $\displaystyle\sum_{n=1}^{\infty} n^3 \left(\tfrac{1}{2}\right)^n$ Looks like geometric series

So compare with a geometric series!

Idea of ratio test:

$\displaystyle\sum_{n=0}^{\infty} ar^n$

$\nearrow$ geometric

$a_n = ar^n$

Terms satisfy:

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{a}\cdot r^{n+1}}{\cancel{a}\cdot r^{n}} = r$$

Series $\sum_{n=0}^{\infty} ar^n$ converges if $|r|<1$.

If series looks geometric try same trick.

e.g. $a_n = \frac{n^3}{2^n}$ $\quad \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^3}{2^{n+1}}}{\frac{n^3}{2^n}} = \frac{2^n}{2^{n+1}} \cdot \frac{(n+1)^3}{n^3}$

$$= \frac{1}{2}\left(\frac{n+1}{n}\right)^3$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^3 = \frac{1}{2}$$

This means that a_n behaves like $\left(\frac{1}{2}\right)^n$ if n is large, so the $\left(\frac{1}{2}\right)^n$ term really dominates the n^3 term.

Ratio test: Given $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$, suppose that

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$. If $\rho < 1$ then series converges. If $\rho > 1$ the series diverges. If $\rho = 1$ we dont know (the test fails).

e.g. $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ we get $\rho = \frac{1}{2}$ so converges.

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

Ratio test. $\displaystyle\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \frac{n!}{(n+1)!} \cdot \frac{10^{n+1}}{10^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \frac{10^{n+1}}{10^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n(n-1)(n-2)\cdots(2)(1)}}{(n+1)\cancel{(n)(n-1)(n-2)\cdots(2)(1)}} \cdot 10$$

$$= \lim_{n\to\infty} \frac{10}{n+1} = 0 \qquad 0<1 \text{ so } \underline{\text{converges}}$$

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \cancel{(n+1)} \cdot \frac{n^n}{(n+1)^n} \cdot \frac{1}{\cancel{n+1}}$$

$$= \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n}$$

$$= \left(1+\frac{1}{n}\right)^{-n} = \Big[\underbrace{\left(1+\frac{1}{n}\right)^n}_{\text{Thm 5}}\Big]^{-1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$$

$$\lim_{n\to\infty} \Big[\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}\Big]^{-1}$$

$$= \frac{1}{e} < 1$$

$\therefore$ $\underline{\text{converges}}$.

2. Root test

<u>Idea</u>: $\sum_{n=0}^{\infty} ar^n$ $\qquad a_n = ar^n$

$$\sqrt[n]{a_n} = (a_n)^{1/n} = (ar^n)^{1/n}$$
$$= a^{1/n} \cdot r$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \underbrace{a^{1/n}}_{1} \cdot r = r.$$

<u>Root test</u>: Given $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$. Suppose that $\lim_{n\to\infty} (a_n)^{1/n} = \rho$. If $\rho < 1$ then series converges. If $\rho > 1$ then series diverges. If $\rho = 1$, then test fails.

<u>e.g.</u> $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ $\qquad$ Root test:

$$(a_n)^{1/n} = \left(\frac{n^3}{2^n}\right)^{1/n} = \frac{n^{3/n}}{2} = \frac{(n^{1/n})^3}{2}$$

$$\lim_{n\to\infty} \frac{(n^{1/n})^3}{2} = \frac{1}{2} < 1 \text{ so } \underline{\text{converges}}.$$

e.g. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{n/2}}$ Root test

$$(a_n)^{1/n} = \left(\frac{n}{(\ln n)^{n/2}}\right)^{1/n} = \frac{n^{1/n}}{[(\ln n)^{n/2}]^{1/n}}$$

$$= \frac{n^{1/n}}{(\ln n)^{1/2}}$$

$$\lim_{n\to\infty} \frac{\overset{\nearrow 1}{n^{1/n}}}{\underset{\searrow \infty}{(\ln n)^{1/2}}} = 0 < 1 \quad \underline{\text{converges}}.$$

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

convergent p-series

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \frac{n^3}{(n+1)^3} = \left(\frac{n}{n+1}\right)^3$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = 1$$

Root test: $\displaystyle\lim_{n\to\infty} (a_n)^{1/n} = 1$ also.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

divergent p-series

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^{1/2}}}{\frac{1}{n^{1/2}}} = \frac{n^{1/2}}{(n+1)^{1/2}} = \left(\frac{n}{n+1}\right)^{1/2}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{1/2} = 1$$

Root test: $\displaystyle\lim_{n\to\infty} (a_n)^{1/n} = 1$ <u>also</u>.

11.2 #9 $\displaystyle\sum_{n=1}^{\infty} \frac{7}{4^n} = \sum_{n=1}^{\infty} 7 \cdot \left(\frac{1}{4}\right)^n$

Geometric series: $\displaystyle\sum_{n=0}^{\infty} ar^n$ or $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$

Then sum is $\dfrac{a}{1-r}$

If series looks like $\displaystyle\sum_{n=1}^{\infty} ar^n = \sum_{n=1}^{\infty} (ar) r^{n-1}$

$r^n = r \cdot r^{n-1}$

$$= \sum_{n=1}^{\infty} 7 \cdot \frac{1}{4} \left(\frac{1}{4}\right)^{n-1} = \sum_{n=1}^{\infty} \underset{a}{\underset{\uparrow}{\frac{7}{4}}} \underset{r}{\underset{\uparrow}{\left(\frac{1}{4}\right)}}^{n-1}$$

$$S = \frac{\frac{7}{4}}{1 - \frac{1}{4}} = \frac{7}{4} \cdot \frac{4}{3} = \frac{7}{3} \,//$$

11.6 Alternating series + Conditional convergence

Idea: $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$ ← so far have assumed this

For series $\sum_{n=1}^{\infty} a_n$ where a_n can be + or −

there are very few techniques. However in one case you can say a lot.

An alternating series has the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n \quad \text{where } u_n \geq 0$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots$$

Theorem: $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges if

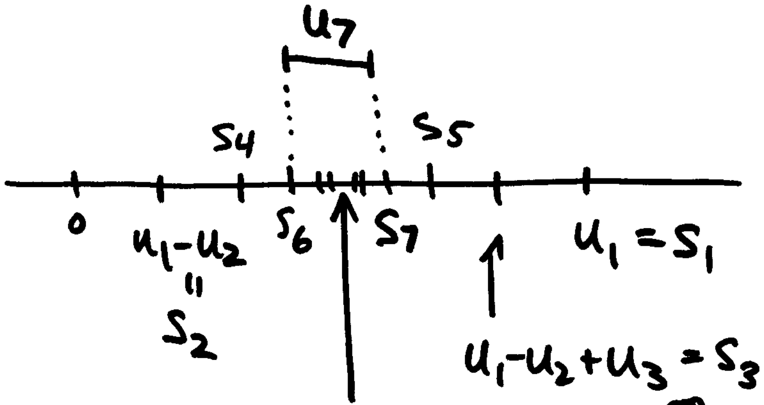
(a) $u_n \geq 0$ for all n,

(b) $u_n \geq u_{n+1}$ ~~for all n~~ eventually, i.e. u_n are decreasing to zero.

(c) $\lim_{n\to\infty} u_n = 0$

Note that it does not matter how fast $u_n \to 0$!

Why? $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots$



$$L = \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n .$$

e.g. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

converges but $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{2/3}}$ converges but $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n)}$ converges.

e.g. $\sum_{n=0}^{\infty} (-1)^{n} \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

converges by Alt. series test. but is also geometric series

So $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$

Compare to

$$\sum_{n=0}^{\infty}\frac{1}{2^n} = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

eg #4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10^n}{n^{10}}$ Diverges because terms do not go to zero.

#6 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{n}$

$\displaystyle\lim_{n\to\infty}\frac{\ln(n)}{n} = 0$

Need to show decreasing to 0

Show: ① $\dfrac{\ln(n)}{n} \geq \dfrac{\ln(n+1)}{n+1}$

$\dfrac{\ln(n)}{\ln(n+1)} \geq \dfrac{n}{n+1}$?

② Derivative:

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x\cdot\frac{1}{x} - \ln(x)\cdot 1}{x^2}$$

$$= \frac{1-\ln x}{x^2} < 0$$

when $\underset{e}{\ln x} > \underset{e}{1}$ or $x > e$

So $\dfrac{\ln(n)}{n}$ is decreasing if $n \geq 3$. This is good enough.

$\therefore$ converges by Alt. Series Test

#10. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}$ Diverges because

$$\lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = 3 \neq 0.$$

Error estimates.

If $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = L$ then $|S_n - L| \le u_{n+1}$

e.g. #46

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n} = \underbrace{\frac{1}{10} - \frac{1}{100} + \frac{1}{1000} - \frac{1}{10000}}_{S_4} + \cdots$$

Then $|S_4 - L| \le u_5 = \frac{1}{10^5} = \frac{1}{100000} = .00001$

2. Absolute convergence

Q: What can you do in general with a series with + and − terms mixed.

e.g. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Series appears to converge. How can I prove this?

Idea: Take absolute values!

Consider
$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$$

series of positive terms.

Theorem: If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

In this case we say $\sum a_n$ converges absolutely.

For above example, use direct comparison

$$\frac{|\cos(n)|}{n^2} \le \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$ converges

and so does $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$.

Series that converge but do not converge absolutely are said to converge conditionally.

e.g. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges by A.S.T.

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right| = \sum_{n>1} \frac{1}{n} \text{ diverges (p-series, } p=1)$$

So $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ does not converge absolutely

and we can say $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges conditionally.

e.g. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ converges by A.S.T.

$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series, $p=2>1$)

$\therefore \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ converges absolutely.

#12) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$ converges absolutely.

$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{(0.1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{(.1)^n}{n}$ converges. by ratio or root

#18) $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin(n)}{n^2}$ converges absolutely.

$\sum_{n=1}^{\infty}\left|(-1)^{n}\frac{\sin(n)}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\sin(n)|}{n^2}$ converges by direct comp to p-series, $p=2$.

#14) $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ converges conditionally.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{1+\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges by limit comparison to p-series, $p = \frac{1}{2} < 1$

Does not converge absolutely.

But it does converge by A.S.T.

#24) $\sum_{n=1}^{\infty} (-1)^{n+1} \sqrt[n]{10}$ diverges because

$\lim_{n \to \infty} \sqrt[n]{10} = \lim_{n \to \infty} 10^{1/n} = 1 \neq 0.$