EXAM 3 SOLUTIONS ON WEB

MAPLE 3 (LAST ONE) IS UP ON-LINE

Looking at series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$

Questions: 1. Does $\sum_{n=1}^{\infty} a_n$ converge?

2. What is its value?

Usually #1 is easier to answer.

As for #2: can find sums of 1 telescoping series and 2 geometric series.

Geometric: $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r|<1$

$$\sum_{k=1}^{n} ar^{k-1} = a \cdot \frac{1-r^n}{1-r} = S_n.$$

1. Series of positive terms.

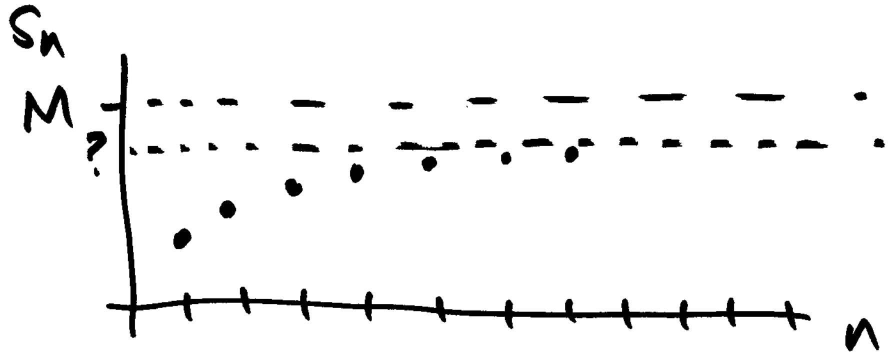
Suppose that for $\sum_{n=1}^{\infty} a_n$, all $a_n \geq 0$.

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

$$S_{n+1} = S_n + \underbrace{a_{n+1}}_{\geq 0} \geq S_n.$$

Conclusion: If $a_n \geq 0$ then $\{s_n\}_{n=1}^{\infty}$ is a non-decreasing sequence where $s_n = \sum_{k=1}^{n} a_k$.

__Fact__: If we can find a number M so that $s_n \leq M$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.



11.3 Integral Test.

Consider $\sum_{n=1}^{\infty} a_n$ where

① $a_n \geq 0$

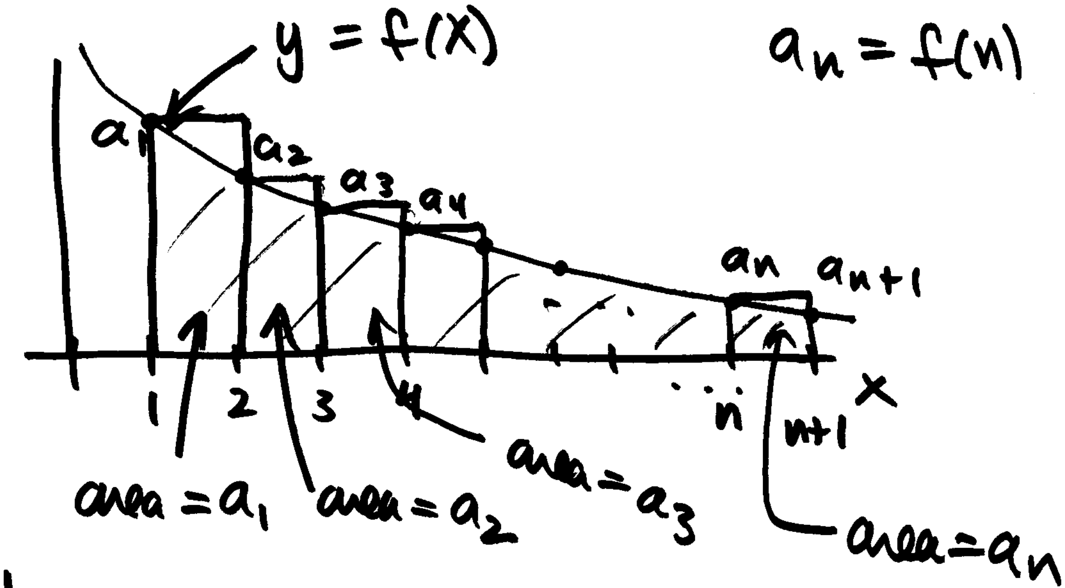
② $a_n = f(n)$ where $f(x)$ is some continuous function

③ $f(x)$ is decreasing as $x \to \infty$ (or "eventually decreasing").

Then $\sum_{n=1}^{\infty} a_n$ converges or diverges with $\int_1^{\infty} f(x)\,dx$.

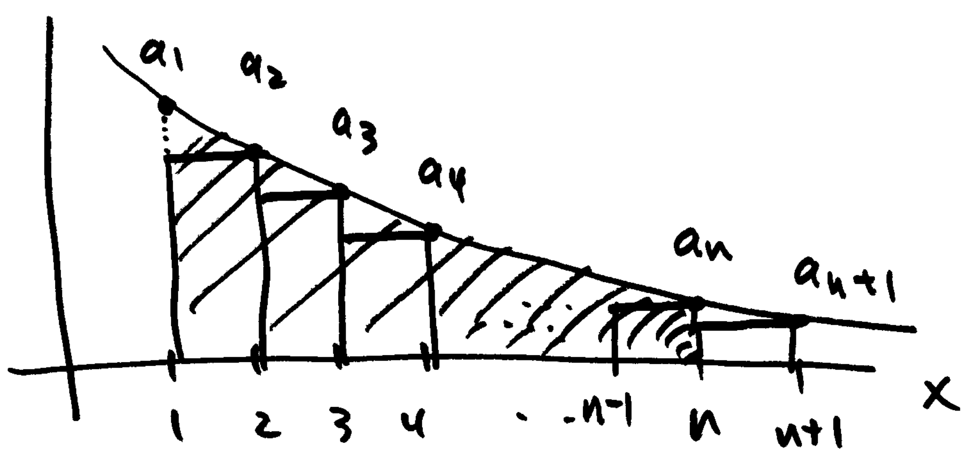
Why?

$a_n = f(n)$



$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k = S_n$$

$$\therefore \text{ If } \int_1^{\infty} f(x)\,dx = \infty \text{ then } \sum_{k=1}^{\infty} a_k = \infty .$$



$$\underbrace{a_1 + a_2 + a_3 + a_4 + \cdots + a_n}_{S_n} \le \int_1^{n} f(x)\,dx + a_1$$

$$S_n = \sum_{k=1}^{n} a_k \le \int_1^{n} f(x)\,dx + a_1 .$$

$\therefore$ If $\int_1^\infty f(x)\,dx$ converges then

$$\sum_{k=1}^{n} a_k \le \underbrace{\int_1^\infty f(x)\,dx + a_1}_{M} < \infty$$

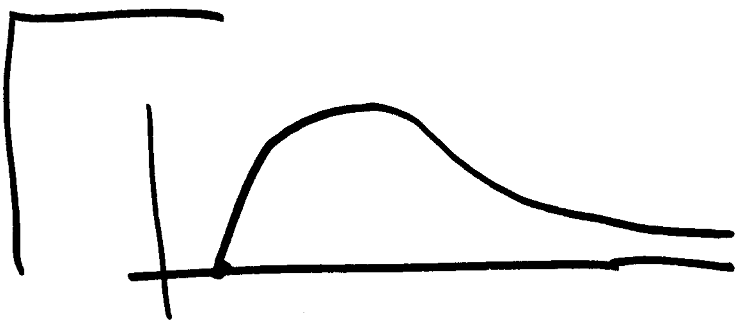
so $\sum_{k=1}^{\infty} a_k$ converges.

Summarize:

$$\boxed{\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k \le \int_1^{n} f(x)\,dx + a_1.}$$

e.g. Harmonic series.

$\sum_{n=1}^{\infty} \frac{1}{n}$ Consider $f(x) = \frac{1}{x}$ decreasing, $\frac{1}{n} = f(n)$.



e.g. $f(x) = \frac{\ln(x)}{x^2}$ eventually decreasing

$f(x) = x^2 e^{-x}$

$$\therefore \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges.}} \text{ converges/diverges with } \underbrace{\int_1^{\infty} \frac{1}{x}\,dx}_{\substack{\text{diverges} \\ \text{since } p=1.}}$$

We can see that it diverges <u>very</u> <u>slowly</u>:

$$\int_1^{b} \frac{1}{x}\,dx = \ln(x)\Big|_1^{b} = \ln(b) - \ln(1) = \ln(b)$$

$$\int_1^{n+1} \frac{1}{x}\,dx \le \sum_{k=1}^{n} \frac{1}{k} \le \int_1^{n} \frac{1}{x}\,dx + 1$$

$$\ln(n+1) \le \sum_{k=1}^{n} \frac{1}{k} \le \ln(n) + 1$$

Q: How many terms do I need to have

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} \ge 20 \ ?$$

Since $\sum_{k=1}^{n} \frac{1}{k} \ge \ln(n+1)$, to guarantee sum is ≥ 20 we can solve $\ln(n+1) \ge 20$

$$\therefore e^{\ln(n+1)} \ge e^{20} \longrightarrow n+1 \ge e^{20} \quad n \ge e^{20} - 1$$

$$e^{20} \approx 500 \text{ million}$$

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p>0$ p-series

By integral test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges/diverges with $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ so

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } 0 \le p \le 1 \end{cases}$$

We can't ~~exact~~ find value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ but we can say:

$$\int_1^{\infty} \frac{1}{x^p}\,dx \le \sum_{n=1}^{\infty} \frac{1}{n^p} \le \int_1^{\infty} \frac{1}{x^p}\,dx + 1$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1}$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx + 1 = \frac{1}{p-1} + 1 = \frac{p}{p-1}$$

e.g. #6 p 759

$$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{-2}{n^{3/2}}$$

$\cancel{f(x) = \frac{-2}{x^{3/2}}}$

~~Actually increasing~~

$$= -2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}}$$

Take $f(x) = \frac{1}{x^{3/2}}$

converges / with $\int_1^{\infty} \frac{1}{x^{3/2}} dx$, $p = \frac{3}{2} > 1$.

OR : Converges because it is a p-series with $p = \frac{3}{2} > 1$.

e.g. #16 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ diverges with $\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx$

$$f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)} = \frac{1}{x+\sqrt{x}}$$

$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx$ diverges with $\int_1^{\infty} \frac{1}{x} dx$ by limit comparison: $\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+\sqrt{x}}} = \lim_{x\to\infty} \frac{x+\sqrt{x}}{x} = 1$.

eg #22) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+(\ln n)^2)}$ converges with $\displaystyle\int_1^{\infty} \frac{1}{x(1+(\ln x)^2)}\,dx$

Integral test: $f(x) = \dfrac{1}{x(1+[\ln(x)]^2)}$

$$\int_1^{\infty} \frac{1}{x(1+\ln(x)^2)}\,dx \qquad = \frac{1}{x+x\ln(x)^2}$$

lim comp with $\frac{1}{x}$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x(1+(\ln x)^2)}} = \lim_{x\to\infty} \frac{\cancel{x}(1+\ln(x)^2)}{\cancel{x}} = \infty$$

DOES NOT WORK

lim-comp. with $\frac{1}{x\ln(x)^2}$.

$$\lim_{x\to\infty} \frac{\frac{1}{x\ln(x)^2}}{\frac{1}{x(1+\ln(x)^2)}} = \lim_{x\to\infty} \frac{\cancel{x}(1+\ln(x)^2)}{\cancel{x}\ln(x)^2} = 1$$

$\displaystyle\int_1^{\infty} \frac{1}{x(1+\ln(x)^2)}\,dx$ conv/ with $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)^2}\,dx$

$$\int_2^\infty \frac{dx}{x\ln(x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln(x)^2}$$

$$\int_2^b \frac{dx}{x\ln(x)^2}$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$x=2 \quad u = \ln(2)$

$x=b \quad u = \ln(b)$

$$= \int_{\ln(2)}^{\ln(b)} \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_{\ln 2}^{\ln b}$$

$$= -\frac{1}{\ln b} + \frac{1}{\ln 2}$$

$$= \lim_{b\to\infty} -\frac{1}{\ln b} + \frac{1}{\ln 2} = \frac{1}{\ln 2} \quad \text{converges.}$$

11.4 Comparison Tests.

1. (Direct) comparison test (Thm 10)

$\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$. If $\sum_{n=1}^{\infty} c_n$ satisfies

1) $a_n \leq c_n$ all n **and** $\sum_{n=1}^{\infty} c_n$ converges

then $\sum_{n=1}^{\infty} a_n$ converges.

2) $c_n \leq a_n$ for all n **and** $\sum_{n=1}^{\infty} c_n$

diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Why? In case 1) $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} c_k \leq \underbrace{\sum_{k=1}^{\infty} c_k}_{<\infty}$

$\therefore$ partial sums are bounded so series converges.

In case 2) $\underbrace{\sum_{k=1}^{n} c_n}_{\to\infty} \leq \sum_{k=1}^{n} a_n$ so also $\sum a_n \to \infty$.

2. Limit Comparison Test. (Thm 11)

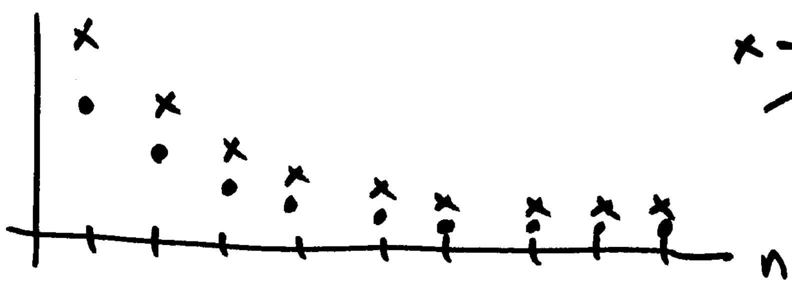
Given $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, $a_n \geq 0$, $b_n \geq 0$.

1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge/diverge together.

2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then if $\sum_{n=1}^{\infty} b_n$ converges so does $\sum_{n=1}^{\infty} a_n$ and if $\sum_{n=1}^{\infty} a_n$ diverges then so does $\sum_{n=1}^{\infty} b_n$.

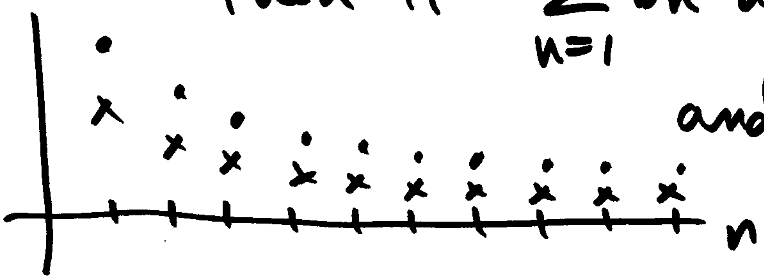
• - a_n

x - b_n



3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

then if $\sum_{n=1}^{\infty} b_n$ diverges so does $\sum_{n=1}^{\infty} a_n$ and if $\sum_{n=1}^{\infty} a_n$ converges so does $\sum_{n=1}^{\infty} b_n$.



x - b_n

• - a_n

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$

Limit comp.: compare with $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{3}{n+\sqrt{n}}} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{3n} = \frac{1}{3}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverge together

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$\left[\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\right]$$

Limit comp:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3+2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+2}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+2}}$$

$$= 1$$

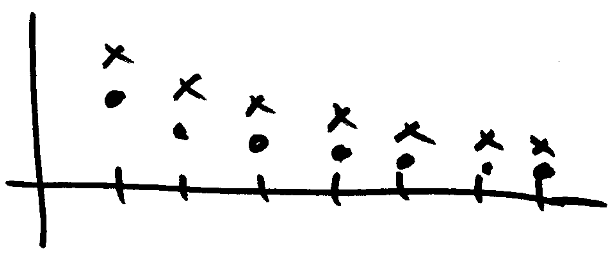
$\therefore$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ converges with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

p-series, $p = \frac{3}{2} > 1$.

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^{3/2}}$ converges by limit comparison test.

Compare with $\dfrac{1}{n^{3/2}}$.

$$\lim_{n\to\infty} \frac{\dfrac{(\ln n)^2}{n^{3/2}}}{\dfrac{1}{n^{3/2}}} = \lim_{n\to\infty} (\ln n)^2 = \infty$$



• – $\dfrac{1}{n^{3/2}}$

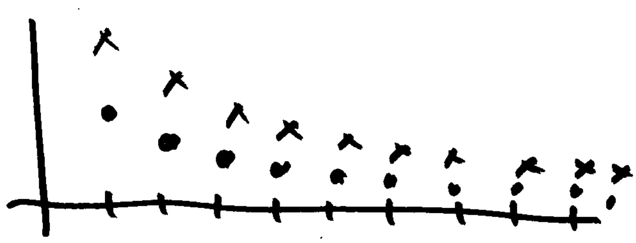
x – $\dfrac{(\ln n)^2}{n^{3/2}}$

NO GOOD.

$$\sum_{n=1}^{\infty} \frac{1}{n^{5/4}} \cdot \boxed{\frac{(\ln n)^2}{n^{1/4}}}$$

Now compare with $\dfrac{1}{n^{5/4}}$

$$\lim_{n\to\infty} \frac{\dfrac{\ln(n)^2}{n^{3/2}}}{\dfrac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{\ln(n)^2}{n^{1/4}} = 0$$



x – $\dfrac{1}{n^{5/4}}$

• – $\dfrac{(\ln n)^2}{n^{3/2}}$