Sequences.

$$\{a_n\}_{n=1}^{\infty} \text{ or } \{a_n\}_{n=0}^{\infty}$$

$$\{a_1, a_2, a_3, a_4, \dots\}$$

Question: What is $\lim_{n\to\infty} a_n$?

Typical approach: Find $f(x)$ so that $f(n)=a_n$, and finding $\lim_{x\to\infty} f(x)$

Theorem 5 p738

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

$6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 720$

Factorials $\longrightarrow$ $n! = n(n-1)(n-2)\cdots(2)(1)$

Exponentials $\longrightarrow$ x^n n changing, x fixed, e.g. 2^n, $(\frac{1}{3})^n$

Polynomials $\longrightarrow$ n^2 $n^{3/2}$ n^5 etc

Logarithms $\longrightarrow$ $\ln(n)$

why? e.g. $\lim_{n\to\infty} \frac{3^n}{n!}$

$$\frac{3^n}{n!} = \frac{3\cdot 3\cdot 3\cdot 3\cdot \cdots \cdot 3}{n\cdot(n-1)(n-2)\cdot(n-3)\cdot\cdots\cdot(1)} = \underbrace{\frac{3}{n}}_{\frac{3}{n}} \cdot \underbrace{\left[\frac{3}{n-1}\cdot\frac{3}{n-2}\cdot\cdots\right]}_{<1} \underbrace{\frac{3}{3}\cdot\frac{3}{2}\cdot\frac{3}{1}}_{(4.5)}$$

$$\therefore \quad \frac{3^n}{n!} \le (13.5)\frac{1}{n} \longrightarrow 0$$

Nothing special about 3^n.

What beats factorials? n^n

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

why?

$$\frac{n!}{n^n} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot(2)(1)}{n \quad n \quad n \quad \cdots \quad n \quad n}$$

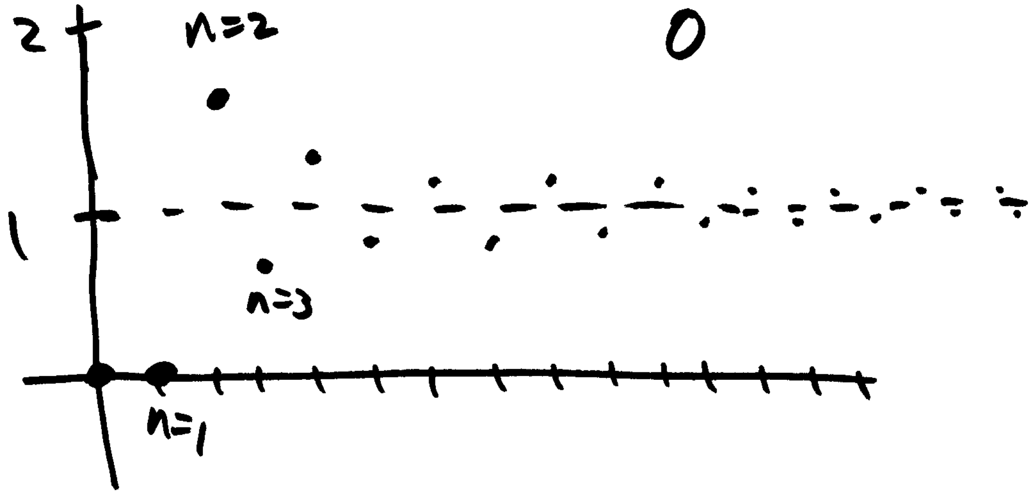
$$= \underbrace{\cancel{\frac{n}{n}}}_{1}\cdot \underbrace{\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\cdots\cdot\frac{2}{n}}_{<1}\cdot\underbrace{\frac{1}{n}}_{\frac{1}{n}}$$

$$\frac{n!}{n^n} \le \frac{1}{n}$$

#24 $a_n = \dfrac{n+(-1)^n}{n}$ Find $\lim\limits_{n\to\infty} a_n$.

$$= \frac{n}{n} + \frac{(-1)^n}{n} = 1 + \underbrace{\cancel{\frac{(-1)^n}{n}}}_{0} \to 1 + 0$$

as $n \to \infty$.



#26 $a_n = \dfrac{2n+1}{1-3\sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{1-3\sqrt{n}} = \lim_{n\to\infty} \frac{2n}{-3\sqrt{n}} = \lim_{n\to\infty} \frac{-2\sqrt{n}}{3}$$
$$= -\infty$$

#38 $a_n = \dfrac{1}{(0.9)^n} \longrightarrow \infty$ as $n \to \infty$

$(0.9)^n \to 0$ as $n \to \infty$.

<u>Also</u> $\dfrac{1}{(.9)^n} = \left(\dfrac{1}{.9}\right)^n = \left(\dfrac{10}{9}\right)^n \to \infty$

since $\frac{10}{9} > 1$.

#42) $a_n = \dfrac{\sin^2 n}{2^n}$

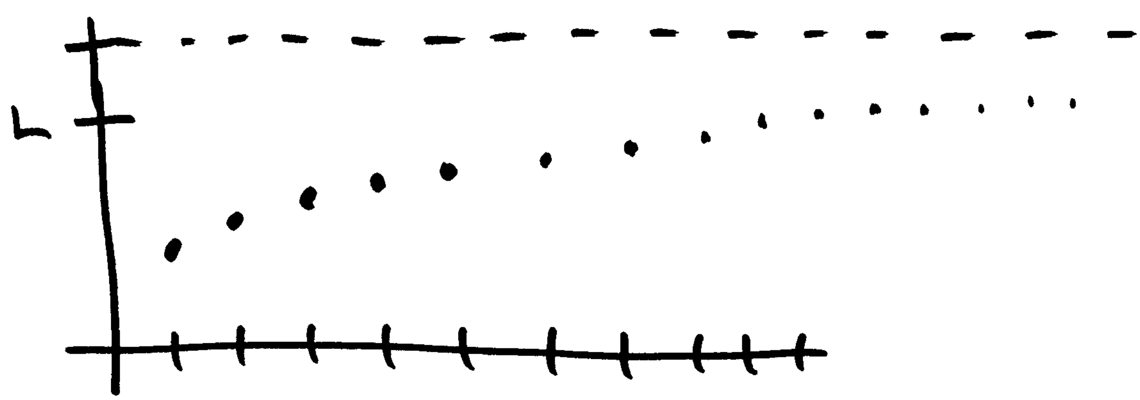
$$\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$$

#58) $a_n = \sqrt[n]{3^{2n+1}}$

$$\lim_{n\to\infty} \left(3^{2n+1}\right)^{1/n} = \lim_{n\to\infty} 3^{\frac{2n+1}{n}} = 3^2 = 9$$

Non-decreasing sequence.

Means $a_{n+1} \geq a_n$ for every n.



Important property: A non-decreasing sequence bounded above always converges.

11.2 Series

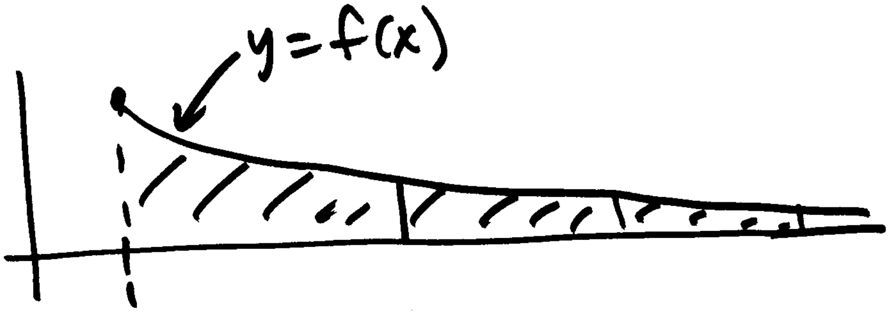
A series is the sum of a sequence.

Given $\{a_n\}_{n=1}^{\infty}$ we write the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

How does this make sense?

Same way that $\int_1^{\infty} f(x)\,dx$ does.



Interpreted $\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \underbrace{\int_1^{b} f(x)\,dx}_{\text{finite integral}}$

Similarly we ~~interpret~~ define:

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$\uparrow$ new index k.

S_n = the n^{th} partial sum of series.

$\{s_n\}_{n=1}^{\infty}$ is the sequence of partial sums.

Then $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

Note: A series $\sum_{n=1}^{\infty} a_n$ is associated to 2 sequences: ① The sequence of terms $\{a_n\}_{n=1}^{\infty}$ ② The sequence of partial sums $\{s_n\}_{n=1}^{\infty}$.

Note: Finding the sum of a series requires knowing a formula for s_n. This is hard in general.

e.g. $$\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

$s_1 = \frac{1}{2}$

$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$s_4 = s_3 + \frac{1}{16} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$

$\vdots$

$s_n = \dfrac{2^n - 1}{2^n}$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n-1}{2^n} = 1$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$$

<u>e.g.</u> #6) $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

$$s_1 = 5\cdot\frac{1}{2}$$

$$s_2 = 5\left(\frac{1}{2}+\frac{1}{6}\right) = 5\cdot\frac{2}{3}$$

$$s_3 = 5\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}\right) = 5\left(\frac{2}{3}+\frac{1}{12}\right) = 5\cdot\frac{3}{4}$$

$$s_4 = 5\left(\frac{3}{4}+\frac{1}{20}\right) = 5\cdot\frac{4}{5}$$

$$s_n = 5\cdot\frac{n}{n+1} \qquad \lim_{n\to\infty} 5\cdot\overbrace{\frac{n}{n+1}}^{1} = 5$$

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} = 5.$$

1. Geometric series

$$\sum_{n=0}^{\infty} ar^n$$

$\downarrow$ or

$$\sum_{n=1}^{\infty} ar^{n-1}$$

e.g. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \Leftrightarrow a = \frac{1}{2}$, $r = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \underset{\substack{\uparrow \\ a}}{\frac{1}{2}} \cdot \underset{\substack{\uparrow \\ r}}{\left(\frac{1}{2}\right)}^{n-1}$$

e.g. $a = 3$, $r = 2$ $\quad \sum_{n=1}^{\infty} 3 \cdot 2^{n-1}$

For $\sum_{n=1}^{\infty} ar^{n-1}$ we have

$$s_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$$

if $r \neq 1$

$$a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\cdots} + \cancel{ar^{n-1}} = s_n$$

$$-(\cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{ar^4} + \cancel{\cdots} + ar^n) = -r s_n$$

$$a - ar^n = s_n - r s_n$$

$$a(1-r^n) = s_n(1-r)$$

$$s_n = \frac{a(1-r^n)}{1-r}.$$

If $|r|<1$ then $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \frac{a(1-r^n)}{1-r}$

$$= \frac{a}{1-r}$$

$\therefore$ if $|r|<1$ then

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

<u>e.g</u> $\displaystyle\sum_{n=1}^{\infty} \underset{\underset{a}{\uparrow}}{\frac{1}{2}} \Big(\underset{\underset{r}{\uparrow}}{\frac{1}{2}}\Big)^{n-1} = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}} = \frac{1}{2}\cdot 2 = 1$

<u>e.g.</u> #10 $\displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot \frac{5}{4^n}$

$$= \sum_{n=0}^{\infty} 5\cdot\frac{(-1)^n}{4^n} = \sum_{n=0}^{\infty} \overset{\overset{a}{\downarrow}}{5}\cdot \Big(\underset{\underset{r}{\uparrow}}{\frac{-1}{4}}\Big)^n$$

$$= \frac{5}{1-\left(\frac{-1}{4}\right)} = \frac{5}{\frac{5}{4}} = 4.$$

e.g. #14

$$\sum_{n=0}^{\infty}\left(\frac{2^{n+1}}{5^n}\right)=\sum_{n=0}^{\infty}\frac{2^n\cdot 2}{5^n}$$

$$=\sum_{n=0}^{\infty}\underbrace{2}_{a}\cdot\underbrace{\left(\frac{2}{5}\right)}_{r}^{n}=\frac{2}{1-\frac{2}{5}}=\frac{2}{\frac{3}{5}}=\frac{10}{3}$$

e.g. #52

$.234234234234\cdots$

$= .234+.000234+.000000234+\cdots$

$$=\frac{234}{1000}+\frac{234}{(1000)^2}+\frac{234}{(1000)^3}+\cdots$$

$$=\sum_{n=0}^{\infty}\underbrace{\left(\frac{234}{1000}\right)}_{a}\underbrace{\left(\frac{1}{1000}\right)}_{r}^{n}$$

$$\sum_{n=1}^{\infty}234\left(\frac{1}{1000}\right)^n=\sum_{n=1}^{\infty}\frac{234}{1000}\cdot\left(\frac{1}{1000}\right)^{n-1}=\sum_{n=0}^{\infty}\frac{234}{1000}\left(\frac{1}{1000}\right)^n$$

$$=\frac{234}{1000}\cdot\frac{1}{1-\frac{1}{1000}}=\frac{234}{1000}\cdot\frac{1000}{999}$$

$$=\frac{234}{999}=\frac{26}{111}$$

e.g. #56 $\quad 1.414141414\cdots$

$$= 1.4 + .014 + .00014 + \cdots$$

$$= \frac{14}{10} + \frac{14}{1000} + \frac{14}{100000} + \cdots$$

$$= \frac{14}{10}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$$

$$\cancel{= \frac{14}{10}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)\right)}$$

$$= \frac{14}{10}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \underset{a}{\underset{\uparrow}{\frac{14}{10}}} \underset{r}{\underset{\uparrow}{\left(\frac{1}{100}\right)}}^{n} = \frac{7}{5} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{7}{\cancel{5}} \cdot \frac{\overset{20}{\cancel{100}}}{99}$$

$$= \frac{140}{99} \quad //$$

2. Telescoping series

e.g. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn \qquad n=0$$

$$\boxed{A=1} \qquad \boxed{B=-1} \qquad n=-1$$

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$S_4 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) = 1 - \frac{1}{5}$$

$$\vdots$$

$$S_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

General form: $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - b_{n+1})$

$$S_n = (b_1 - \cancel{b_2}) + (\cancel{b_2} - \cancel{b_3}) + (\cancel{b_3} - \cancel{b_4}) \cancel{+} \cdots \cancel{+} (\cancel{b_n} - b_{n+1})$$
$$= b_1 - b_{n+1}$$

3. Test for divergence

Fact: If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Why?

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k$$

$$= \cancel{a_1} + \cancel{a_2} + \cancel{a_3} + \cdots \cancel{} + a_n$$
$$- (\cancel{a_1} + \cancel{a_2} + \cancel{a_3} + \cdots + \cancel{a_{n-1}}) = a_n$$

$$a_n = S_n - S_{n-1}$$

So if $\displaystyle\lim_{n\to\infty} S_n = S$ then $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$
$$= S - S = 0.$$

e.g. $\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+1-1 \cdots$

$\stackrel{?}{=} 0$?

$\lim_{n\to\infty} (-1)^n$ DNE. so $\sum_{n=1}^{\infty} (-1)^n$ cant converge.

what is going on?

$S_1 = -1$

$S_2 = 0$

$S_3 = -1$

$S_4 = 0$

$\vdots$

$\lim_{n\to\infty} S_n$ does not exist

eg #36) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ Does not ~~exist~~ converge.

because $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$.

Note: If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ does not necessarily converge. Need $a_n \to 0$

FAST ENOUGH!