Review for Exam 3:

8.5 § 3) $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

$\boxed{\begin{array}{ll} \sqrt{a^2-x^2} & x = a\sin\theta \\ \sqrt{a^2+x^2} & x = a\tan\theta \\ \sqrt{x^2-a^2} & x = a\sec\theta \end{array}}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$4+x^2 = 4+4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$

$x=2 \quad 2 = 2\tan\theta \to \tan\theta = 1 \to \theta = \frac{\pi}{4}$

$x=-2 \quad -2 = 2\tan\theta \to \tan\theta = -1 \to \theta = -\frac{\pi}{4}$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\sec^2\theta}{4\sec^2\theta}\, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\, d\theta = \frac{1}{2}\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{4}$$ //

7) $\displaystyle\int \sqrt{25-t^2}\, dt$

$t = 5\sin\theta$

$dt = 5\cos\theta\, d\theta$

$\sqrt{25-t^2} = \sqrt{25-25\sin^2\theta}$

$= \sqrt{25(1-\sin^2\theta)} = 5\sqrt{1-\sin^2\theta}$

$= 5\cos\theta$

$\theta = \sin^{-1}\left(\frac{t}{5}\right)$

$= \displaystyle\int 5\cos\theta \cdot 5\cos\theta\, d\theta$

$= 25\displaystyle\int \cos^2\theta\, d\theta$

$= 25\displaystyle\int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$

$= 25\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$

$$= 25\left(\underbrace{\tfrac{1}{2}\sin^{-1}\left(\tfrac{t}{5}\right) + \tfrac{1}{4}\sin\left(2\cdot\sin^{-1}\left(t/5\right)\right)}\right) + C$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right)$$

$$+ \frac{1}{50}t\sqrt{25-t^2} + C$$

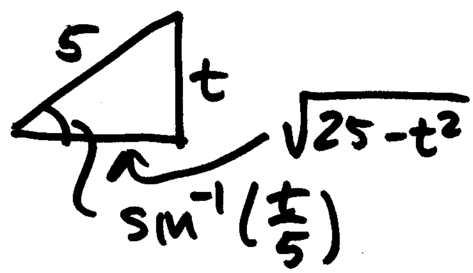
Trig identity:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\sin\left(2\cdot\sin^{-1}\left(\frac{t}{5}\right)\right)$$

$$= 2\sin\left(\sin^{-1}\left(\frac{t}{5}\right)\right)\cdot\cos\left(\sin^{-1}\left(\frac{t}{5}\right)\right)$$

$$= 2\cdot\frac{t}{5}\cdot\frac{1}{5}\sqrt{25-t^2}$$



$$= \frac{2}{25}t\sqrt{25-t^2}$$

27) $$\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$$

$$v = \sin\theta$$

$$dv = \cos\theta\,d\theta$$

$$1-v^2 = 1-\sin^2\theta = \cos^2\theta$$

$$(1-v^2)^{5/2} = (\cos^2\theta)^{5/2} = \cos^5\theta$$

$$v^2 = \sin^2\theta$$

$$\theta = \sin^{-1}(v)$$

$$= \int \frac{\sin^2\theta\cdot\cos\theta}{\cos^5\theta}\,d\theta$$

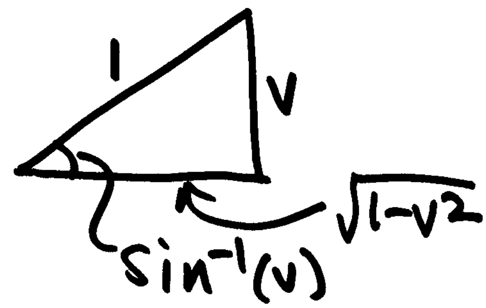
$$= \int \frac{\sin^2\theta}{\cos^4\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\cos^2\theta}\, d\theta = \int \tan^2\theta \sec^2\theta \, d\theta$$

$$\boxed{\begin{aligned} u &= \tan\theta \\ du &= \sec^2\theta \, d\theta \end{aligned}} \qquad = \int u^2 \, du = \frac{1}{3}u^3 + c$$

$$= \frac{1}{3}\tan^3\theta + c = \frac{1}{3}\tan^3(\sin^{-1}(v)) + c$$

$$= \frac{1}{3}\left[\tan(\sin^{-1}(v))\right]^3 + c$$



$$= \frac{1}{3}\left[\frac{v}{\sqrt{1-v^2}}\right]^3 + c$$

$$= \frac{1}{3}\,\frac{v^3}{(1-v^2)^{3/2}} + c \quad //$$

23) $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

$$x = \sin\theta$$
$$dx = \cos\theta \, d\theta$$
$$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$$
$$(1-x^2)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$$

$$= \int \frac{\cos^3\theta \cdot \cos\theta}{\sin^6\theta}\, d\theta$$

$$= \int \frac{\cos^4\theta}{\sin^6\theta}\, d\theta = \int \frac{\cos^4\theta}{\sin^4\theta} \cdot \frac{1}{\sin^2\theta}\, d\theta$$

$$= -\int \cot^4\theta \cdot -\csc^2\theta \, d\theta \qquad u = \cot\theta$$

$$du = -\csc^2\theta \, d\theta$$

$$= -\int u^4 \, du \quad \text{etc} \cdots$$

29) $\displaystyle\int_0^{\ln 4} \frac{e^t \, dt}{\sqrt{e^{2t}+9}}$

$u = e^t$

$du = e^t \, dt$

$e^{2t} + 9 = u^2 + 9$

$t = 0 \quad u = 1$

$t = \ln 4 \quad u = 4$

$$= \int_1^4 \frac{du}{\sqrt{u^2+9}}$$

$u = 3\tan\theta$

etc

$\vdots$

8.6 29) $\displaystyle\int \frac{\sqrt{3t-4}}{t} \, dt$

#12 with $a = 3 \quad b = -4$

$$= 2\sqrt{3t-4} - 4\underbrace{\int \frac{dt}{t\sqrt{3t-4}}}_{\#13(a) \quad a=3, \; b=4}$$

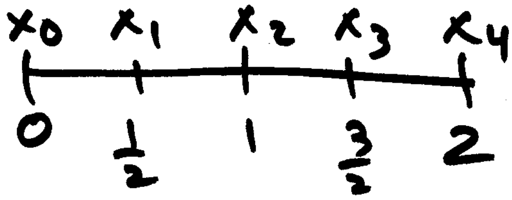
$$= 2\sqrt{3t-4} - 4\left[\frac{\cancel{2}}{\cancel{\sqrt{4}}} \tan^{-1}\left(\sqrt{\frac{3t-4}{4}}\right)\right] + C$$

$$= 2\sqrt{3t-4} - 4\tan^{-1}\left(\sqrt{\frac{3t-4}{4}}\right) + C \,.$$

8.7 5) $\displaystyle\int_0^2 t^3 + t \, dt = \frac{1}{4}t^4 + \frac{1}{2}t^2 \Big|_0^2$

$$= 4 + 2 - 0 = 6 \; //$$

Trap: $n = 4$



$\Delta x = \frac{1}{2}$

$y_0 = x_0^3 + x_0 = 0$

$y_1 = (\frac{1}{2})^3 + \frac{1}{2} = \frac{5}{8}$

$y_2 = 2$

$y_3 = \frac{27}{8} + \frac{3}{2} = \frac{39}{8}$

$y_4 = 10$

$$\int_0^2 t^3 + t \, dt$$

$$\approx \frac{\Delta x}{2}\left(0 + 2\cdot\frac{5}{8} + 2\cdot 2 + 2\cdot\frac{39}{8} + 10\right)$$

$$= \frac{1}{4}\left(\frac{5}{4} + 4 + \frac{39}{4} + 10\right)$$

$$= \frac{1}{4}\left(\frac{5+16+39+40}{4}\right) = \frac{1}{4}\cdot\frac{100}{4} = \frac{25}{4}$$

$$= 6.25$$

$$E_T = |6.25 - 6| = .25$$

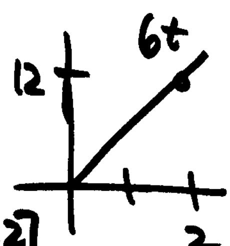
$f(t) = t^3 + t$

$f'(t) = 3t^2 + 1$

$f''(t) = 6t$

$|f''(t)| \le 12$ on $[0,2]$

$\therefore M = 12$



$$|E_T| \le \frac{M(b-a)^3}{12n^2} = \frac{12\cdot 2^3}{12\cdot 4^2} = \frac{1}{2}$$

$\underline{\text{Simpson}}$: $\frac{\Delta x}{3}\left(0+4\cdot\frac{5}{8}+2\cdot 2+4\cdot\frac{39}{8}+10\right)$

$\Delta x = \frac{1}{2}$

$$= \frac{1}{6}\left(\frac{5}{2}+4+\frac{39}{2}+10\right)$$

$$= \frac{1}{6}\left(\frac{5+8+39+20}{2}\right) = \frac{1}{6}\cdot\frac{72}{2} = \frac{36}{6} = 6$$

$E_S = 0$!

$\underline{\text{exact!}}$

$\underline{8.8}$ 25) $\int_0^1 x\ln x\,dx = \lim_{b\to 0^+}\int_b^1 x\ln x\,dx$

$\int_b^1 x\ln x\,dx$ $\quad u=\ln(x)\quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v=\frac{1}{2}x^2$

$$= \frac{1}{2}x^2\ln x\Big|_b^1 - \frac{1}{2}\int_b^1 x\,dx$$

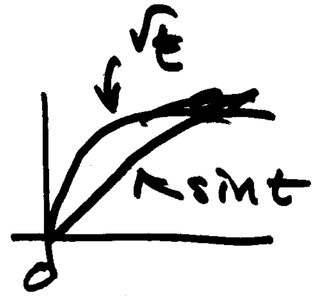
$$= \cancelto{0}{\frac{1}{2}\ln 1} - \frac{1}{2}b^2\ln b - \frac{1}{2}\left(\frac{1}{2}x^2\Big|_b^1\right)$$

$$= -\frac{1}{2}b^2\ln b - \frac{1}{4} + \frac{1}{4}b^2$$

$$= \lim_{b\to 0^+} \cancelto{0}{\frac{1}{4}b^2} - \frac{1}{4} - \cancelto{0}{\frac{1}{2}b^2\ln b} = -\frac{1}{4}$$

41) $\displaystyle\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t} = \lim_{b\to 0^+} \int_b^{\pi} \frac{dt}{\sqrt{t}+\sin t}$

$$= \int_0^{\pi} \frac{dt}{\sqrt{t}\left(1+\frac{\sin t}{\sqrt{t}}\right)}$$



Use Limit comparison:
Compare with $\frac{1}{\sqrt{t}}$

$$\lim_{t\to 0^+} \frac{\frac{1}{\sqrt{t}}}{\frac{1}{\sqrt{t}+\sin t}}$$

$$= \lim_{t\to 0^+} \frac{\sqrt{t}+\sin t}{\sqrt{t}} = \lim_{t\to 0^+} 1 + \frac{\sin t}{\sqrt{t}} = 1$$

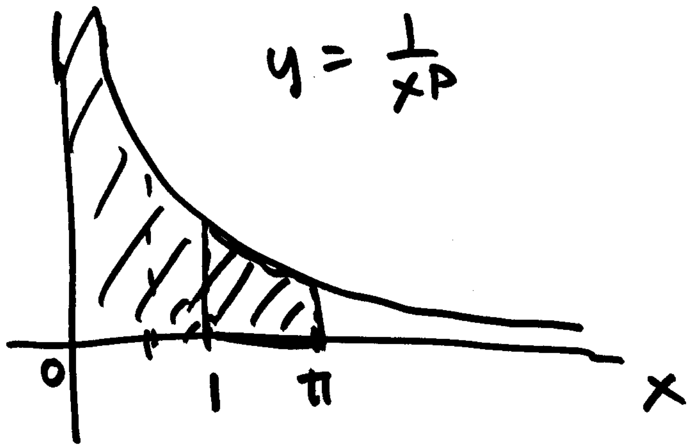
$$\frac{\sin t}{\sqrt{t}} = \underbrace{\sqrt{t}}_{\to 0}\underbrace{\left(\frac{\sin t}{t}\right)}_{\to 1} \to 0 \quad \text{as } t\to 0$$

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

$\therefore \displaystyle\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$ converges with $\displaystyle\int_0^{\pi} \frac{dt}{\sqrt{t}}$

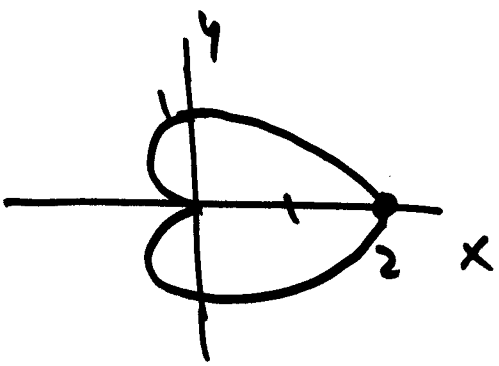
p-integrals:

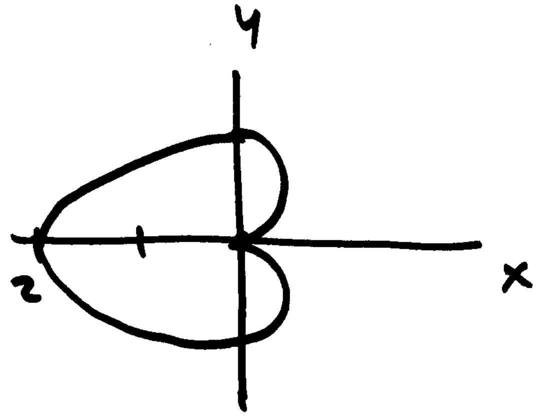
$$\int_1^{\infty} \frac{1}{x^p}dx = \begin{cases} \infty & \text{if } 0<p\le 1 \\ \frac{1}{p-1} & \text{if } p>1 \end{cases} \qquad \int_0^{1} \frac{1}{x^p}dx = \begin{cases} \infty & \text{if } p\ge 1 \\ \frac{1}{1-p} & \text{if } 0<p<1 \end{cases}$$

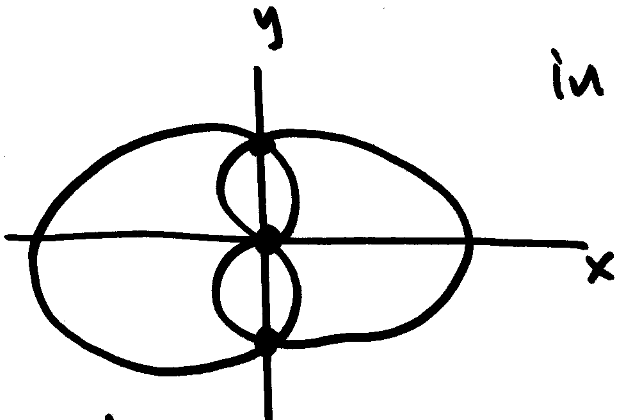


10.6 31) $r = 1 + \cos\theta$

$r = 1 - \cos\theta$







intersect at:

$(0,0)$ origin

$(0,1)$

$(0,-1)$

Algebraically

$1 + \cos\theta = 1 - \cos\theta$

$2\cos\theta = 0$

$\cos\theta = 0$

$\therefore \theta = \frac{\pi}{2}, -\frac{\pi}{2}$, etc...

So at $r = 1, \theta = \frac{\pi}{2}$

$r = 1, \theta = -\frac{\pi}{2}$

Note: For $r = 1 + \cos\theta$, $r = 0$ when $\theta = \pi$

So $(0, \pi)$ [in polar] is on graph.

$r = 1 - \cos\theta$ $r = 0$ when $\theta = 0$

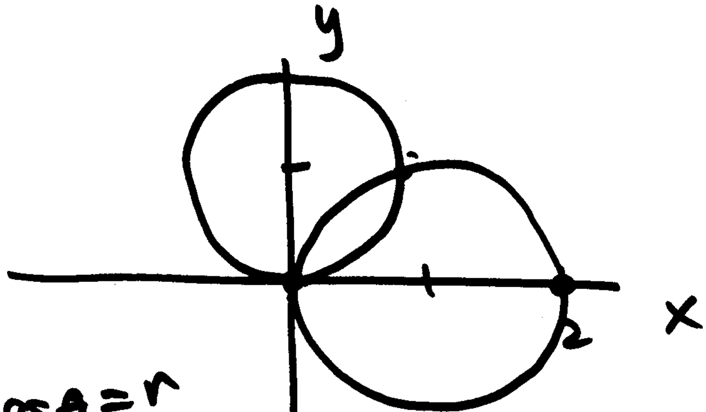
So $(0, 0)$ [in polar] is on graph

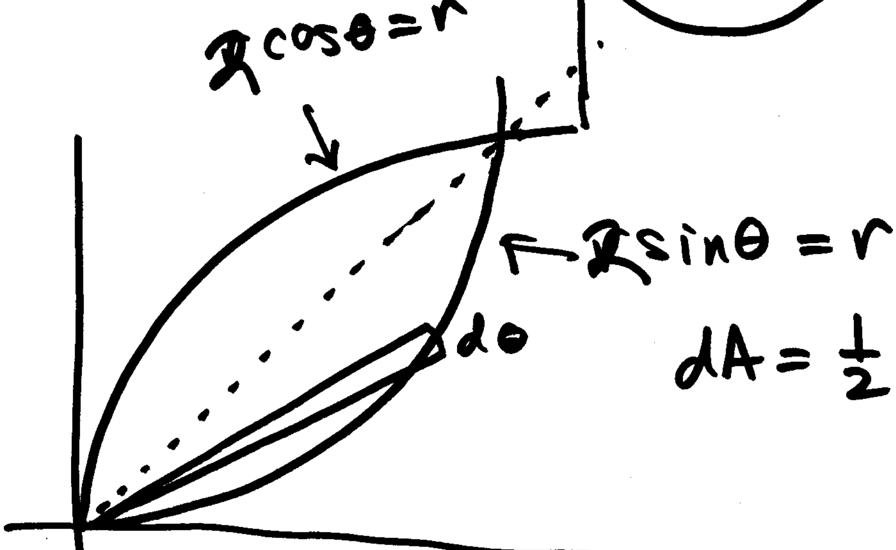
But $(0, \pi) = (0, 0)$ (both are the origin)

10.7 7) $r = 2\cos\theta$

$r = 2\sin\theta$

Picture





$dA = \frac{1}{2} r^2 \, d\theta$

Intersection point:

$$2\cos\theta = 2\sin\theta$$

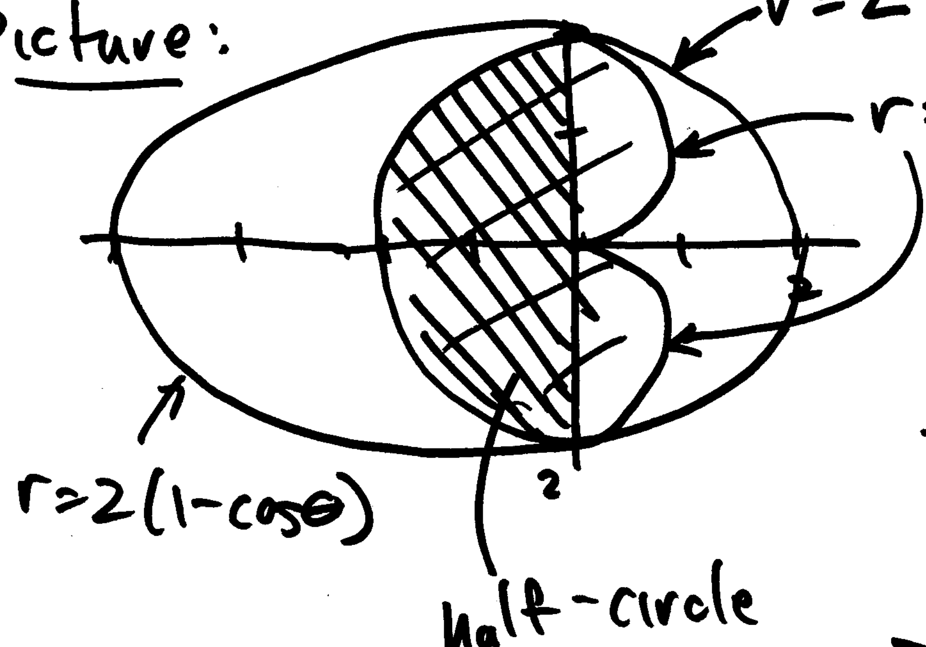
$$\cos\theta = \sin\theta \quad \therefore \theta = \frac{\pi}{4}$$

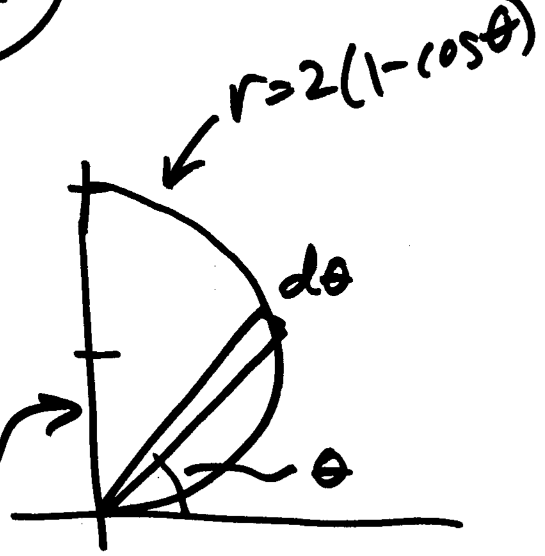
$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(2\sin\theta)^2\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4\sin^2\theta\, d\theta \quad \text{etc}\ldots$$

9) $r=2 \qquad r=2(1-\cos\theta)$

Picture:





$$\text{Total area} = \begin{pmatrix}\text{Area of}\\ \text{half circle}\end{pmatrix} + 2\cdot\begin{pmatrix}\text{Area}\\ \text{of}\\ \text{bump}\end{pmatrix}$$

$$= \frac{1}{2}\pi(2)^2 + 2\int_0^{\pi/2} \frac{1}{2}\left[2(1-\cos\theta)\right]^2 d\theta$$

$$= 2\pi + \int_0^{\pi/2} 4(1-\cos\theta)^2 \, d\theta$$

etc...

23) $r = \dfrac{\cancel{2}6}{1+\cos\theta} \qquad 0 \le \theta \le \dfrac{\pi}{2}$

$$L = \int_\alpha^\beta \left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{1/2} d\theta$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(6(1+\cos\theta)^{-1}\right)$$

$$= 6\left(-(1+\cos\theta)^{-2}(-\sin\theta)\right)$$

$$= \frac{6\sin\theta}{(1+\cos\theta)^2}$$

$$= \int_0^{\pi/2} \left(\frac{36}{(1+\cos\theta)^2} + \frac{36\sin^2\theta}{(1+\cos\theta)^4}\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \frac{6}{1+\cos\theta}\left(1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \frac{6}{1+\cos\theta} \left(\frac{(1+\cos\theta)^2 + \sin^2\theta}{(1+\cos\theta)^2} \right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \frac{6}{1+\cos\theta} \cdot \left(\frac{1 + 2\cos\theta + \overbrace{\cos^2\theta + \sin^2\theta}^{1}}{(1+\cos\theta)^2} \right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \frac{6}{1+\cos\theta} \left(\frac{2\,\cancel{(1+\cos\theta)}}{(1+\cos\theta)^{\cancel{2}}} \right)^{1/2} d\theta$$

$$= 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^{3/2}}$$

$$\cos^2(x) = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)$$

$$2\cos^2(x) = 1 + \cos(2x)$$

$$= 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{(2\cos^2(\frac{\theta}{2}))^{3/2}}$$

$$= 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{2^{3/2} \cos^3(\frac{\theta}{2})}$$

$$= \frac{6\sqrt{2}}{2\sqrt{2}} \int_0^{\pi/2} \frac{d\theta}{\cos^3(\frac{\theta}{2})} = 3 \int_0^{\pi/2} \sec^3\left(\frac{\theta}{2}\right) d\theta$$

This is doable.