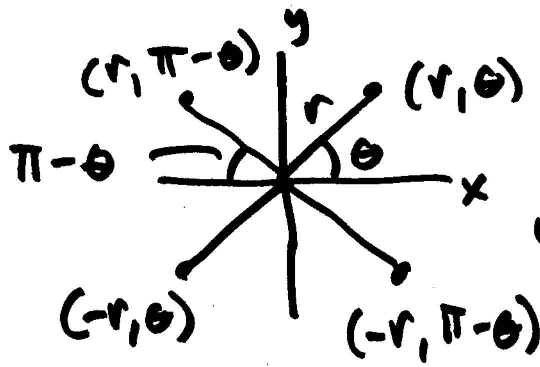
MAPLE 2 due tomorrow.

EXAM 3 Friday on 8.5–8.8, 10.5–10.7, 11.1

Polar graphs (cont'd)

<u>e.g.</u> $r^2 = \sin\theta$

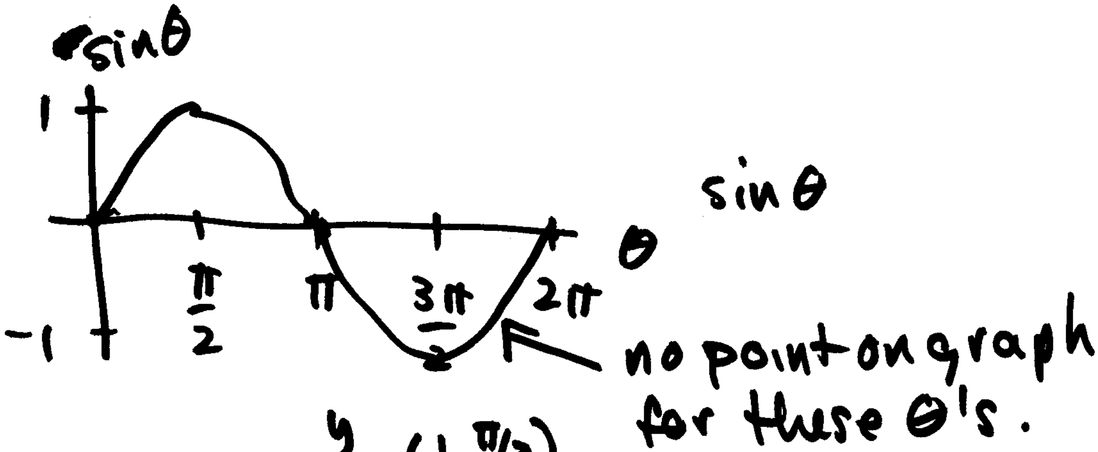
Symmetries: $\sin\theta$ odd means <u>y-axis symm.</u>

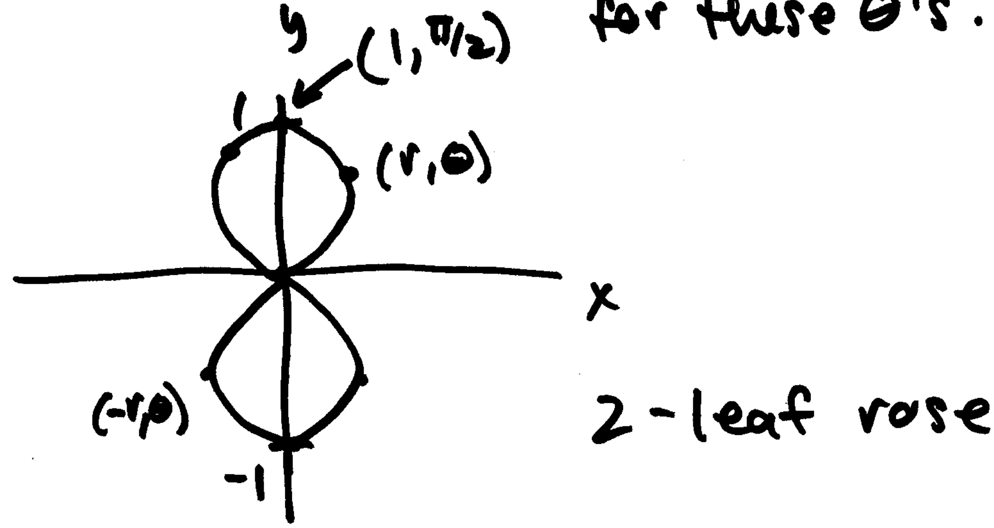


Since $(-r)^2 = r^2$ we know that (r,θ) on graph $\Rightarrow$ $(-r,\theta)$ on graph. This is symm. about <u>origin</u>.

y-axis symm + origin symmetry gives <u>x-axis</u> symmetry

Graph.





2. Slopes of polar graphs.
(Nothing new.) $r = f(\theta)$

$$\left.\begin{aligned} x &= r\cos\theta = f(\theta)\cos\theta \\ y &= r\sin\theta = f(\theta)\sin\theta \end{aligned}\right\}$$ parametric equations with parameter θ

Rcl: Slopes of parametric curves.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)}$$

$$= \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

eg #18) $r = -1 + \sin\theta$ $\quad\quad$ $\theta = 0, \pi$

$$\cancel{\frac{dy}{d\theta}} = \quad y = r\sin\theta = \sin\theta(-1+\sin\theta)$$

$$x = r\cos\theta = \cos\theta(-1+\sin\theta)$$

$$\frac{dy}{d\theta} = \sin\theta\cos\theta + \cos\theta(-1+\sin\theta)$$

$$= 2\sin\theta\cos\theta - \cos\theta$$

$$\frac{dx}{d\theta} = \cos^2\theta - \sin\theta(-1+\sin\theta)$$

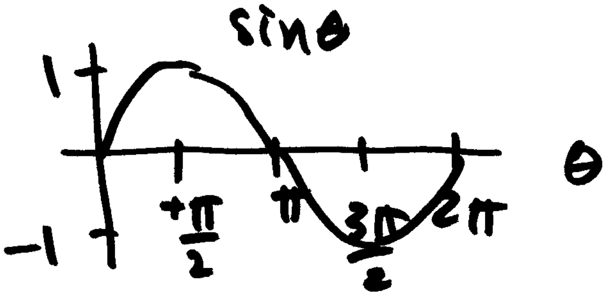
$$= \cos^2\theta - \sin^2\theta + \sin\theta$$

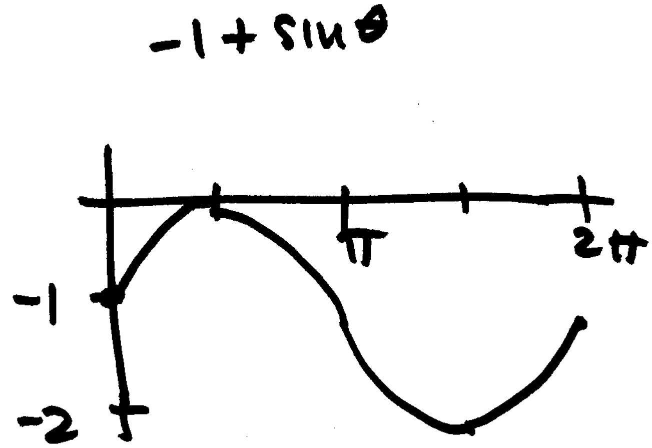
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin\theta\cos\theta - \cos\theta}{\cos^2\theta - \sin^2\theta + \sin\theta}$$

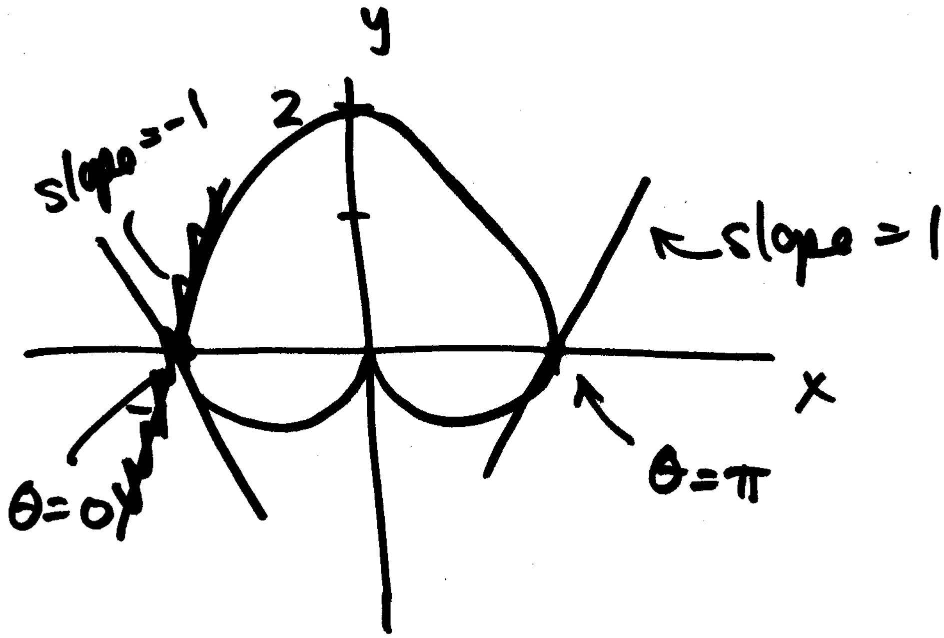
$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{-1}{1} = -1$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{+1}{1} = 1$$

Sketch: $r = -1 + \sin\theta$

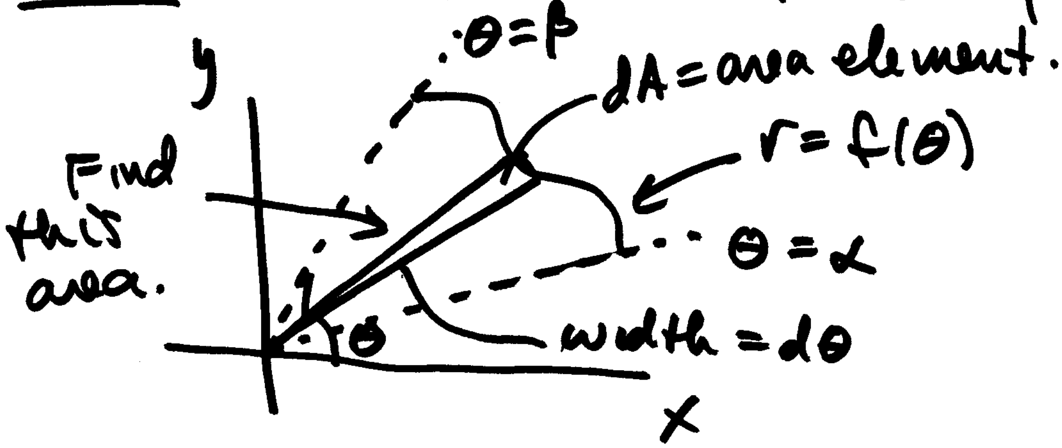






10.7 Areas + Lengths

Areas: $r = f(\theta) \qquad \alpha \le \theta \le \beta$



Divide $[\alpha, \beta]$ into small subintervals of width $d\theta$ at θ.

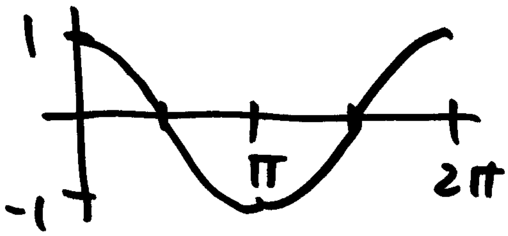
$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} f(\theta)^2 d\theta$$

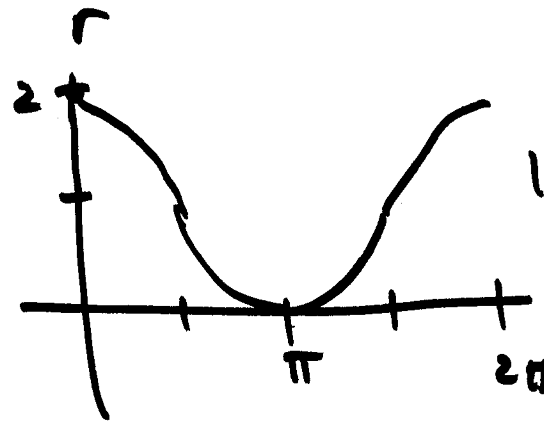
$$\text{Area} = \underbrace{\pi r^2}_{\substack{\text{area}\\\text{of full}\\\text{circle}}} \cdot \underbrace{\frac{d\theta}{2\pi}}_{\substack{\text{portion}\\\text{of circle}\\\text{corresp to}\\\text{wedge } d\theta.}} = \frac{1}{2} r^2 d\theta$$

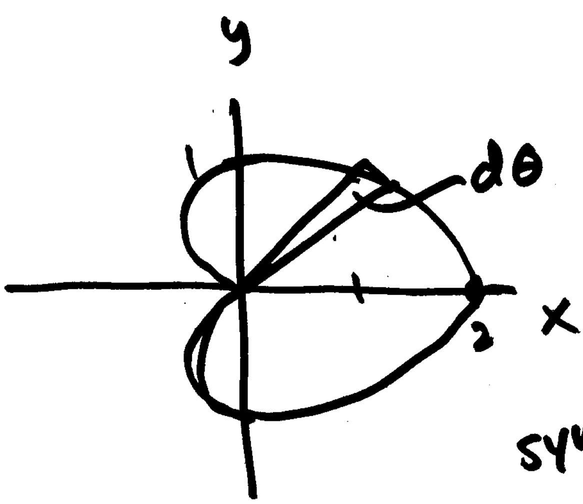
$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta.$$

e.g. #2 p714 Take $a=1$

$$r = 1 + \cos\theta$$







$$dA = \frac{1}{2} r^2 \, d\theta$$
$$= \frac{1}{2}(1+\cos\theta)^2 \, d\theta$$

symmetry

$$A = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta = 2\int_0^{\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta$$

$$= \int_0^{\pi} (1+\cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \;\; d\theta$$

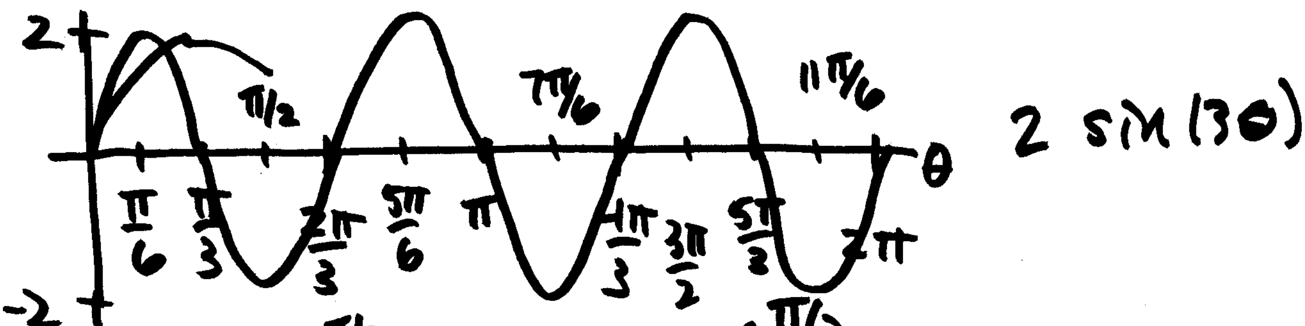
$$= \int_0^{\pi} 1 + 2\cos\theta + \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$$

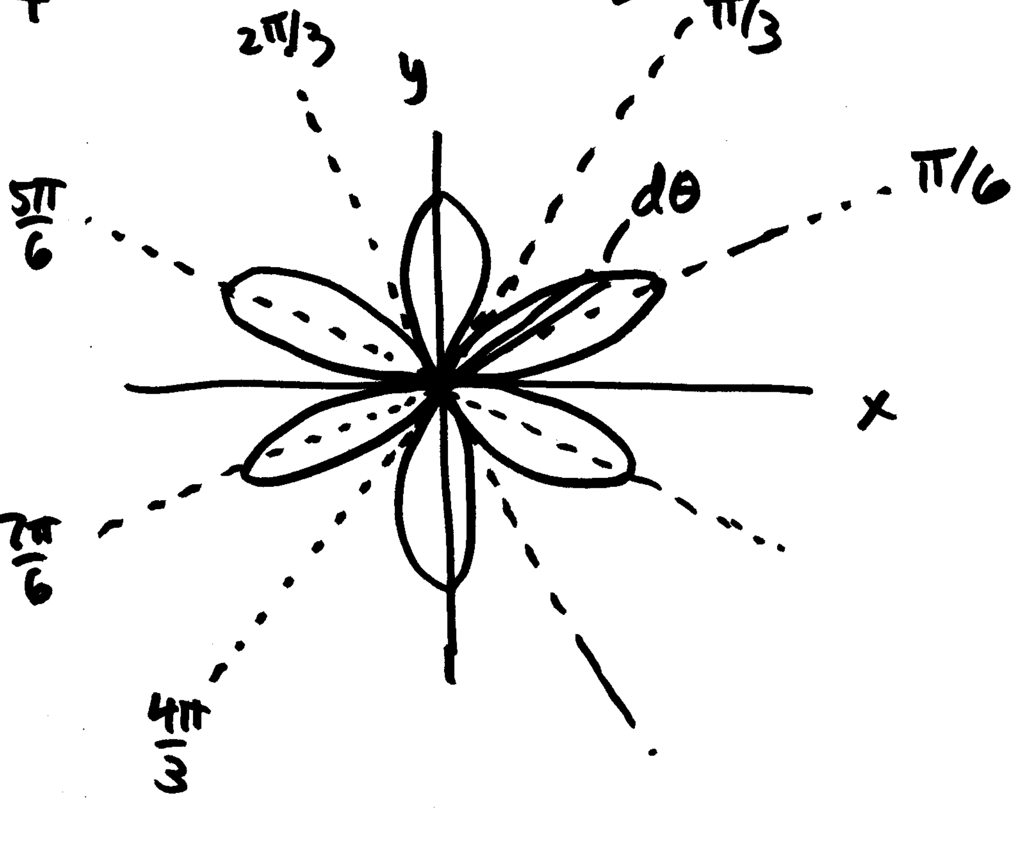
$$= \int_0^{\pi} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos(2\theta)\, d\theta$$

$$= \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta) \Big|_0^{\pi}$$

$$= \frac{3}{2}\pi \quad //$$

<u>e.g.</u> #6) $r^2 = 2\sin(3\theta)$





Want <u>area</u>. Want area of one leaf.

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}(2\sin(3\theta))^2 \, d\theta = \sin 3\theta \, d\theta$$

$$\underbrace{A = \int_0^{\frac{\pi}{3}} \sin 3\theta \, d\theta,}_{} \quad \text{Total area} = 6\int_0^{\frac{\pi}{3}} \sin 3\theta \, d\theta$$

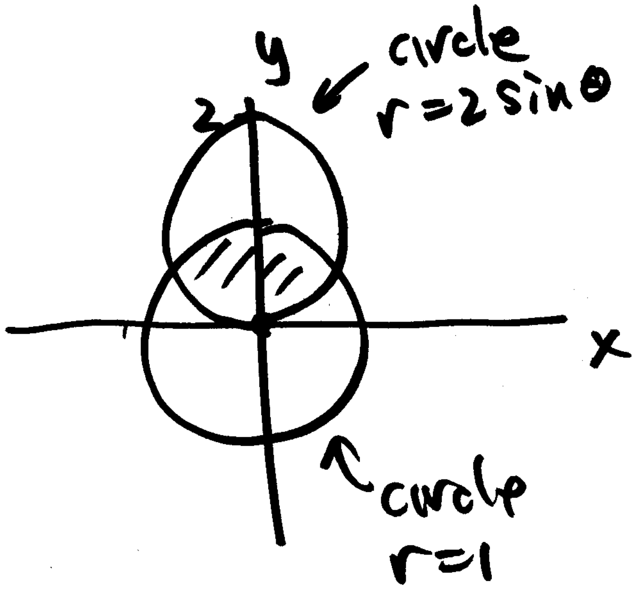
↑ one leaf

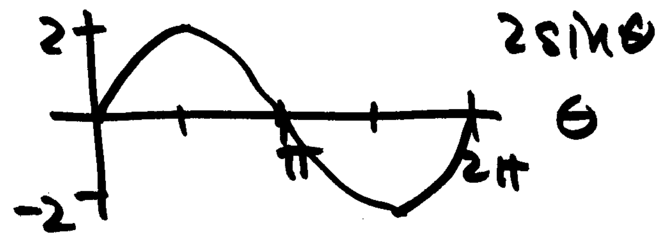
$$= 6 \cdot \frac{-1}{3} \cos(3\theta) \Big|_0^{\pi/3}$$

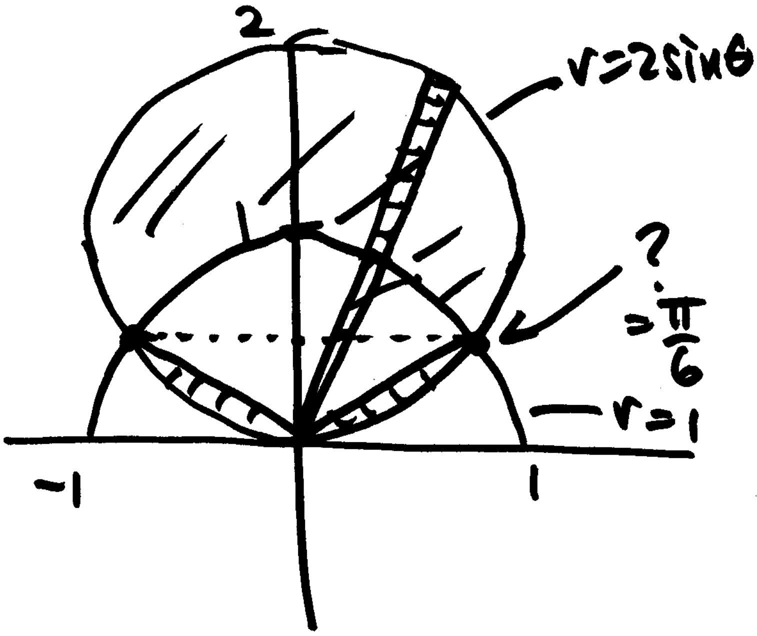
$$= -2(\cos(\pi) - \cos(0))$$

$$= -2(-1-1) = 4 \quad //$$

#8) $r=1 \quad r = 2\sin\theta$







$1 = 2\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

<u>Method 1</u>: $\underbrace{\text{Area of full circle}}$ — $\underbrace{\text{Area of cap.}}$

$$dA = \frac{1}{2}\left((2\sin\theta)^2 - 1^2\right)d\theta$$

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(4\sin^2\theta - 1\right)d\theta$$

$$\text{area} = \frac{1}{2}r_2^2\,d\theta - \frac{1}{2}r_1^2\,d\theta$$

$$= \frac{1}{2}\left(r_2^2 - r_1^2\right)d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\pi/2} \frac{1}{2}\left(4\sin^2\theta - 1\right)d\theta$$

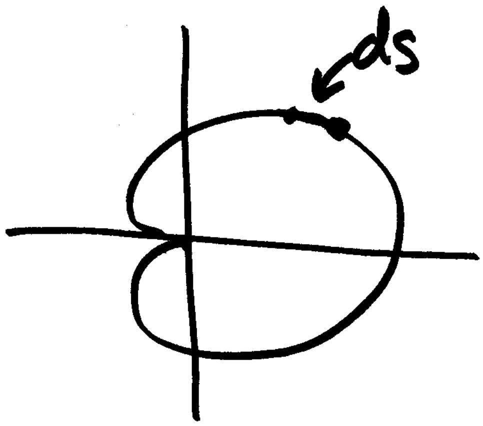
Symmetry

$$\text{Total Area} = \pi - \cancel{2}\int_{\pi/6}^{\pi/2}\left(4\sin^2\theta - 1\right)d\theta \quad \text{etc...}$$

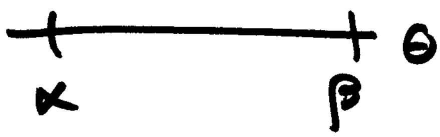
2. Lengths of Polar graphs.

$r = f(\theta)$ $\qquad$ $x = f(\theta)\cos\theta$

$\qquad\qquad\qquad\;\; y = f(\theta)\sin\theta$



$$ds = (dx^2 + dy^2)^{1/2}$$
$$= \left(\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2 \right)^{1/2} d\theta$$



Lenth of curve:

$$L = \int_\alpha^\beta ds = \int_\alpha^\beta \left(\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2 \right)^{1/2} d\theta$$

This simplifies:

From before $\quad \dfrac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$$

$$\left[\frac{dx}{d\theta}\right]^2 = \left(-f(\theta)\sin\theta + f'(\theta)\cos\theta\right)^2$$

$$f(\theta)^2\sin^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta$$
$$+ f'(\theta)^2\cos^2\theta$$

$$\left[\frac{dy}{d\theta}\right]^2 = \left(f(\theta)\cos\theta + f'(\theta)\sin\theta\right)^2$$
$$= f(\theta)^2\cos^2\theta + 2f(\theta)f'(\theta)\cos\theta\sin\theta$$
$$+ f'(\theta)^2\sin^2\theta$$

$$\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2 = f(\theta)^2 + f'(\theta)^2$$

$$= \int_\alpha^\beta \left(f(\theta)^2 + f'(\theta)^2\right)^{1/2} d\theta = \int_\alpha^\beta \left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{1/2} d\theta$$

e.g. #22 $a=1$

$r = \sin^2(\theta/2) \qquad 0 \le \theta \le \pi$

Find length of curve.

$$L = \int_0^\pi \left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{1/2} d\theta$$

$$\frac{dr}{d\theta} = \cancel{2}\sin(\theta/2)\cdot\cos(\theta/2)\cdot\frac{1}{\cancel{2}}$$

$$L = \int_0^{\pi} \left(\sin^4\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \right)^{1/2} d\theta$$

$$= \int_0^{\pi} \left[\sin^2\left(\frac{\theta}{2}\right) \underbrace{\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} \right)}_{1} \right]^{1/2} d\theta$$

$$= \int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= -2\cos\left(\frac{\theta}{2}\right)\Big|_0^{\pi}$$

$$= -2\cancelto{0}{\cos\frac{\pi}{2}} + 2\cos(0) = 2$$

11.1 Sequences

Idea: $f(x)$ = polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_n x^n$$

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

$$a_0 = f(0)$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$$

$$f'(0) = a_1 \qquad \underline{x=0}$$

$$f''(x) = 2a_2 + 6a_3 x + \cdots + n(n-1) a_n x^{n-2}$$

$$a_2 = \frac{1}{2} f''(0) \qquad \underline{x=0}$$

In general: $a_n = \frac{1}{n!} f^{(n)}(0)$

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0)\, x^k$$

Q: What if $f(x)$ <u>not</u> a polynomial?
e.g. $f(x) = \sin(x)$?

Can we write

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) x^k \text{ ?}$$

for non-polynomial $f(x)$?

called <u>Taylor series</u>

In Ch. 11 we look at ∞ sums like this

First we look at sums of <u>numbers</u>

of the form $\sum_{n=0}^{\infty} a_n$ <u>a_n numbers</u>

Before this want to look at <u>sequences</u>

of numbers $\{a_n\}_{n=0}^{\infty}$.

What is a sequence?

A <u>sequence</u> is a function whose domain is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ <u>or</u> $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$

e.g. $f(n) = n^2$

We usually write: $a_n = n^2$

or $\{a_1, a_2, a_3, a_4, \cdots\}$

$= \{1, 4, 9, 16, 25, 36, \cdots\}$

e.g. $f(n) = \dfrac{1}{2n+1}$ n in $\mathbb{N}$.

$a_n = \dfrac{1}{2n+1}$ $\{a_1, a_2, a_3, a_4, \cdots\}$

$= \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots\}$

Also write $\{a_n\}_{n=1}^{\infty}$

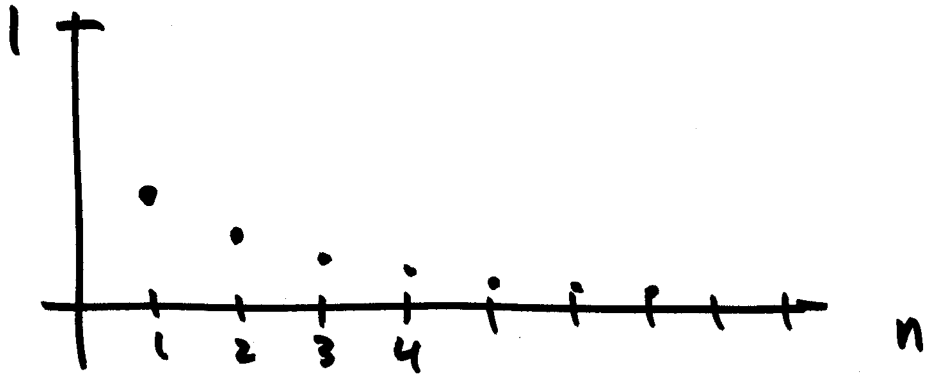
e.g. $f(n) = (\frac{1}{2})^n$ $a_n = (\frac{1}{2})^n$

$\{(\frac{1}{2})^n\}_{n=1}^{\infty}$ $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots\}$

Also write $\{\frac{1}{2^n}\}_{n=1}^{\infty}$

Sometimes graph a sequence.

$$a_n = \frac{1}{2n+1}$$



Sometimes sequences are defined recursively.

eg #10 $\quad \underline{a_1 = -2}$

$$a_{n+1} = \frac{n a_n}{n+1}$$

$$a_2 = \frac{1 \cdot a_1}{2} = \frac{1}{2} a_1 = -1$$

$$a_3 = \frac{2 \cdot a_2}{3} = \frac{2}{3} a_2 = -\frac{2}{3}$$

$$a_4 = \frac{3 \cdot a_3}{4} = \frac{3}{4} a_3 = \frac{\cancel{3}}{\cancel{4}_2} \cdot \frac{-\cancel{2}}{\cancel{3}} = -\frac{1}{2}$$

$$a_5 = \frac{4 \cdot a_4}{5} = \frac{4}{5} a_4 = \frac{\cancel{4}^2}{5} \cdot \frac{-1}{\cancel{2}} = -\frac{2}{5}$$

etc...

e.g. Fibonacci numbers

$$a_1 = 1$$
$$a_2 = 1$$

$$a_{n+1} = a_n + a_{n-1}$$

$$a_3 = a_2 + a_1 = 2$$
$$a_4 = a_3 + a_2 = 3$$
$$a_5 = a_4 + a_3 = 5$$
$$a_6 = a_5 + a_4 = 8$$

etc...

Main concern is what happens to a_n as $n \to \infty$?

That is: Find $\lim_{n \to \infty} a_n$.

To do this you use same ideas, same tools as with finding

$$\lim_{x \to \infty} f(x).$$

Thm 4 p 7836 important

If $a_n = f(n)$ for some function $f(x)$ defined on real numbers, then

$$\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n .$$

e.g. $a_n = n^2$ $\quad n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} x^2 = \infty$$

e.g. $a_n = \frac{1}{2n+1}$ $\quad f(x) = \frac{1}{2x+1}$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{1}{2x+1} = 0$$

e.g. $a_n = \frac{2n^2+n+1}{n^2-5}$ $\quad n = 1, 2, \ldots$

$f(x) = \frac{2x^2+x+1}{x^2-5}$ $\quad a_n = f(n)$ for $n \geq 3$

So $$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{2x^2+x+1}{x^2-5} = 2$$

Thm 5 p 738 VERY IMPORTANT

#2. $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = 1$

$$\ln\left[\lim_{n\to\infty} n^{\frac{1}{n}}\right] = \lim_{n\to\infty} \ln(n^{1/n})$$

$$= \lim_{n\to\infty} \frac{1}{n}\ln(n) = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

3. $\lim_{n\to\infty} x^{1/n} = x^0 = 1$ if $x \neq 0$.

4. $\lim_{n\to\infty} x^n = 0$ if $|x|<1$

[and $= \infty$ if $|x|>1$]

e.g. $x=2$

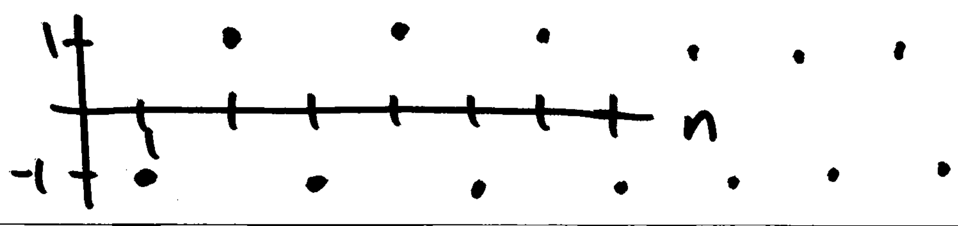
$\lim_{n\to\infty} 2^n = \infty$

$x = \frac{1}{3}$ $\lim_{n\to\infty} (\frac{1}{3})^n = 0$

[if $x=1$

$\lim_{n\to\infty} 1^n = 1$]

$\lim_{n\to\infty} (-1)^n$



Does not exist

$$\lim_{n\to\infty} a_n = L$$