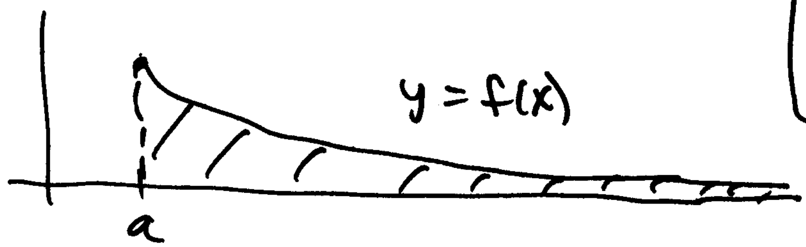
MAPLE #2 due Thursday

EXAM #3 Friday 8.5–8.8, 10.5–10.7, 11.1–11.2 ?

Improper Integrals (cont'd)



Unbounded region. Finite or Infinite area?

$\int_a^\infty f(x)\,dx$ = area.

In all cases we will encounter,

$\lim_{x\to\infty} f(x) = 0.$

Area being finite or infinite depends on <u>how fast</u> f(x) decays to zero as $x \to \infty$.

<u>e.g.</u> $\int_0^\infty e^{-x}\,dx < \infty$ Exponential decay is fast enough for finite area

$\int_0^\infty x^2 e^{-x}\,dx < \infty$ Any polynomial × an exponential is fast enough.

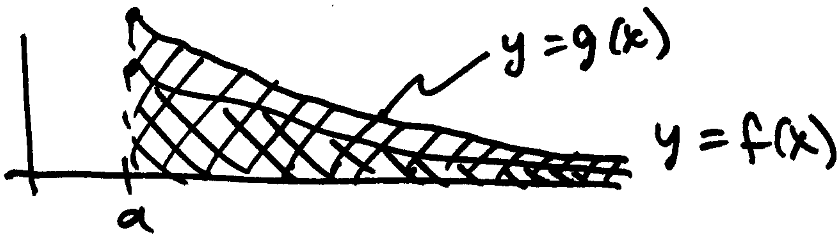
$\int_1^\infty \frac{1}{x^p}\,dx,\ p>0$ $\quad p>1 \Rightarrow$ area is finite; $\quad p \le 1 \Rightarrow$ area is infinite

$\int_1^\infty \frac{\ln(x)}{x^2}\,dx < \infty$ Any power of ln(x) times $\frac{1}{x^p}$, p>1 decays fast enough.

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx}$$

Convergence tests. Allows you to determine convergence or not without computing integral.

1. Direct comparison.



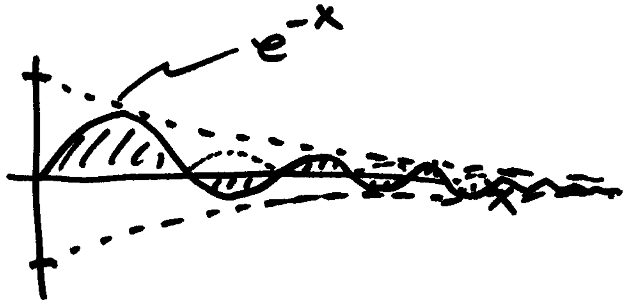
$\int_a^\infty f(x)\,dx$ converge? ⓐ Spse $0 \le f(x) \le g(x)$ for all x <u>and</u> $\int_a^\infty g(x)\,dx < \infty$. Then $\int_a^\infty f(x)\,dx < \infty$. ⓑ Spse $0 \le f(x) \le g(x)$ for all x <u>and</u> $\int_a^\infty f(x)\,dx = \infty$ (i.e. it diverges). Then $\int_a^\infty g(x)\,dx = \infty$ (i.e. it diverges also).

<u>e.g.</u> $\displaystyle\int_1^\infty \frac{dx}{x^4+1}$ Compare with $\frac{1}{x^4}$

want: $\displaystyle\frac{1}{x^4+1} \le \frac{1}{x^4}$ <u>True</u>

$\displaystyle\therefore \int_1^\infty \frac{dx}{x^4+1} \le \int_1^\infty \frac{dx}{x^4} < \infty$ by p-integral test since $p = 4 > 1$

e.g. $\displaystyle\int_0^\infty e^{-x}\sin(x)\,dx$



Compare with e^{-x}

Want: $|e^{-x}\sin(x)| \le e^{-x}$ Yes.

why? $|e^{-x}\sin(x)| = e^{-x}|\sin(x)|$

$\le e^{-x}$ since $|\sin(x)| \le 1$.

By Direct Comparison, $\displaystyle\int_0^\infty e^{-x}\sin(x)\,dx$ converges.

e.g. $\displaystyle\int_2^\infty \frac{dx}{x^4-1}$ Still behaves like $\frac{1}{x^4}$, so compare:

Want $\displaystyle\frac{1}{x^4-1} \le \frac{1}{x^4}$ True? NO

In fact $\displaystyle\frac{1}{x^4-1} \ge \frac{1}{x^4}$. What to do?

① Try $\frac{1}{x^3}$ $\displaystyle\frac{1}{x^4-1} \overset{?}{\le} \frac{1}{x^3}$ ✗ ✓

$x^3 \le x^4 - 1$

$1 \le x^4 - x^3 = x^3(x-1)$ ✓

True since for $x \ge 2$, $x-1 \ge 1$ and $x^3 \ge 1$

② Try say $\frac{2}{x^4}$

$$\frac{1}{x^4-1} \overset{?}{\leq} \frac{2}{x^4} \checkmark \quad \underline{\text{True}}$$

$$x^4 \leq 2(x^4-1)$$

$$x^4 \leq 2x^4-2$$

$$2 \leq 2x^4-x^4 = x^4 \checkmark$$

Another approach:

2. Limit comparison test:

Spse $f(x), g(x) \geq 0$ for $x \geq a$, are continuous and that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L > 0$. Then

$\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

~~For~~ e.g. $\int_2^\infty \frac{dx}{x^4-1}$

compare with $\frac{1}{x^4}$

$$\lim_{x\to\infty} \frac{\frac{1}{x^4}}{\frac{1}{x^4-1}} = \lim_{x\to\infty} \frac{x^4-1}{x^4} =$$

$$\lim_{x\to\infty} \frac{x^4}{x^4} = 1 > 0$$

$\therefore$ by Limit comp. $\int_2^\infty \frac{dx}{x^4-1}$ converges with $\int_2^\infty \frac{1}{x^4}\,dx$.

__e.g.__ $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$ Behaves like: $\dfrac{1}{x\sqrt{x^2}} = \dfrac{1}{x^2}$

Try limit comp:

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{x\sqrt{x^2-1}}} = \lim_{x\to\infty} \frac{x\sqrt{x^2-1}}{x^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\underbrace{x}_{\sqrt{x^2}}} = \lim_{x\to\infty} \sqrt{\frac{x^2-1}{x^2}}$$

$$= \sqrt{\lim_{x\to\infty}\left(\frac{x^2-1}{x^2}\right)} = \sqrt{1} = 1 > 0$$

By limit comp $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$ converges with $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ (p-int. with $p=2>1$)

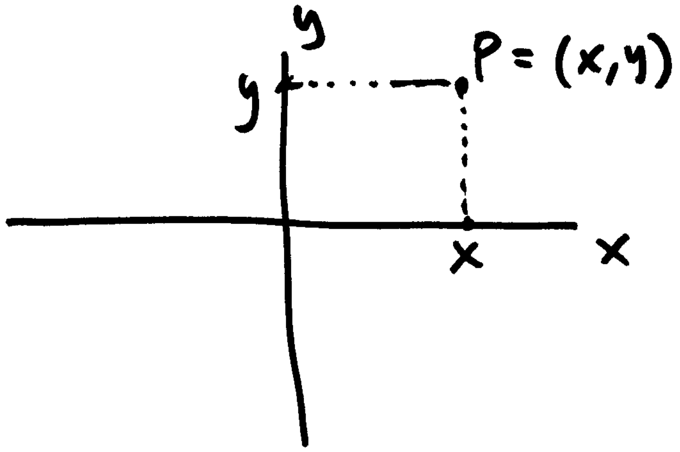
__e.g.__ $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}-1}$ Behave like: $\dfrac{1}{\sqrt{x}}$

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}-1}} = \lim_{x\to\infty} \frac{\sqrt{x}-1}{\sqrt{x}}$$

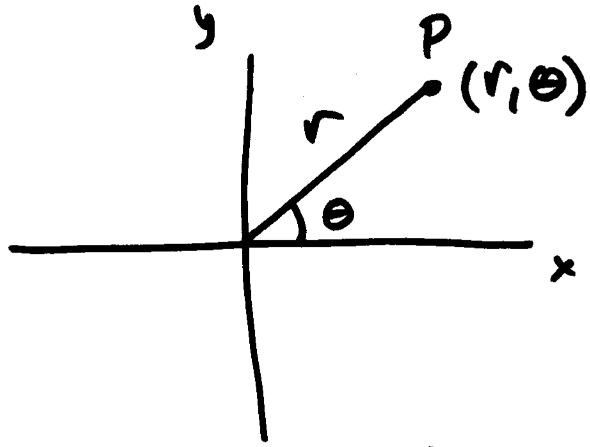
$$= \lim_{x\to\infty}\left(1 - \frac{1}{\sqrt{x}}\right) = 1$$

So by lim-comp. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}-1}$ diverges with $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ (p-int. with $p=\frac{1}{2}<1$).

10.5 Polar coordinates



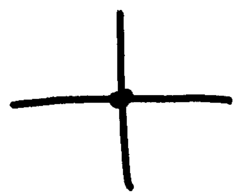
Cartesian coordinates
(rectangular coordinates)



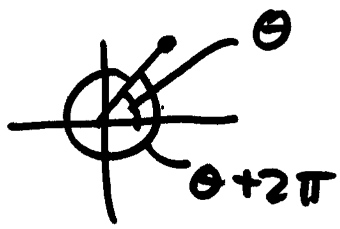
Polar coordinates
r, θ real numbers
θ = radian measure of angle.

1. Non-uniqueness

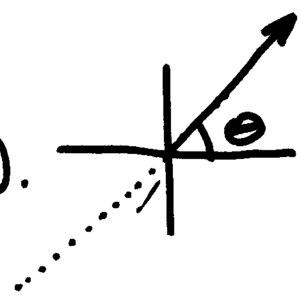
a. $(0, \theta)$ represents the origin for any θ.



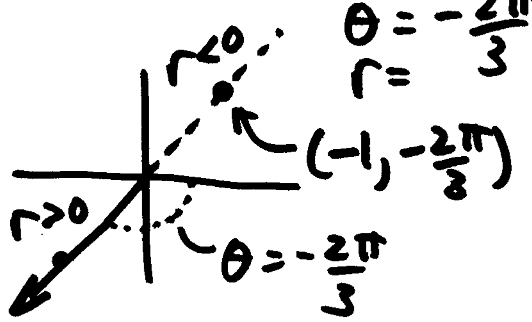
b. $(r, \theta) = (r, \theta + 2\pi)$



c. $(r, \theta) = (-r, \theta + \pi)$
$= (-r, \theta - \pi)$.

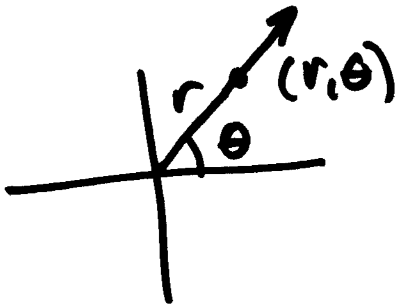


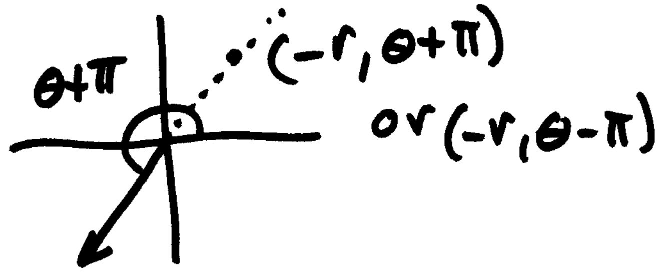
$\theta = -\frac{2\pi}{3}$
$r =$



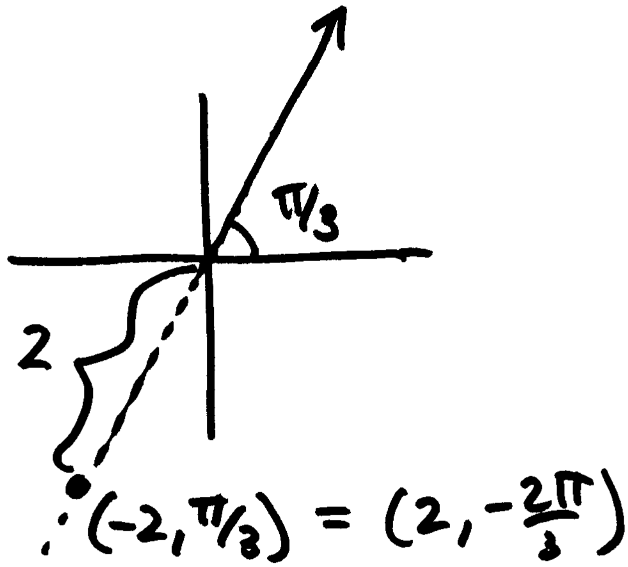
(r, θ)

Think of θ as a "direction"
$r > 0$ means move r in that direction, $r < 0$ means move $|r|$ in opposite direction

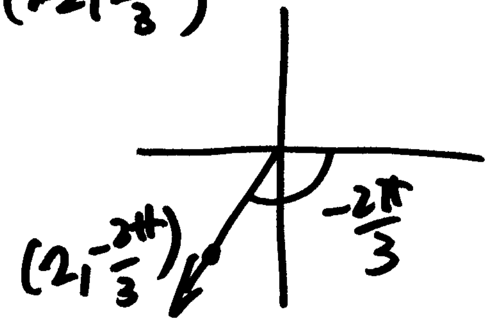




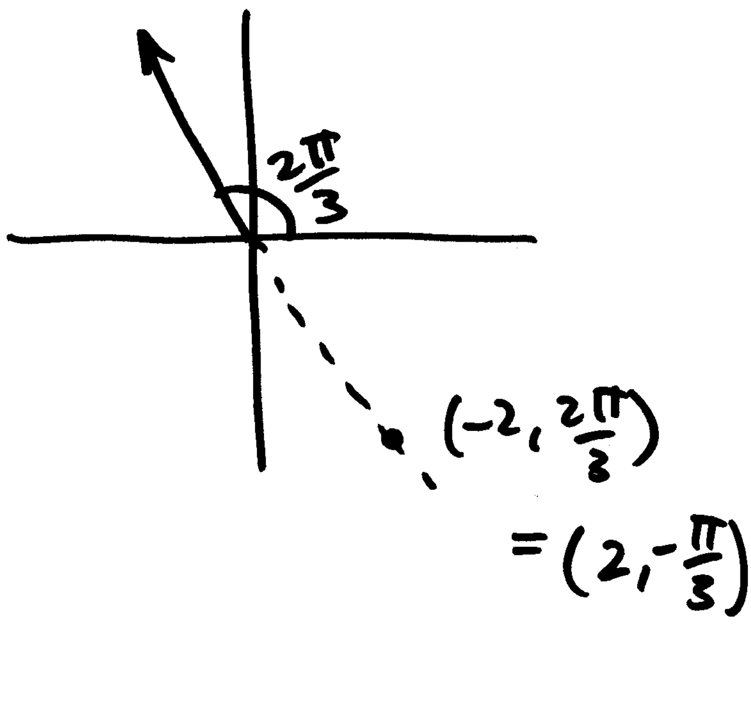
eg #2) $\left(-2, \frac{\pi}{3}\right)$

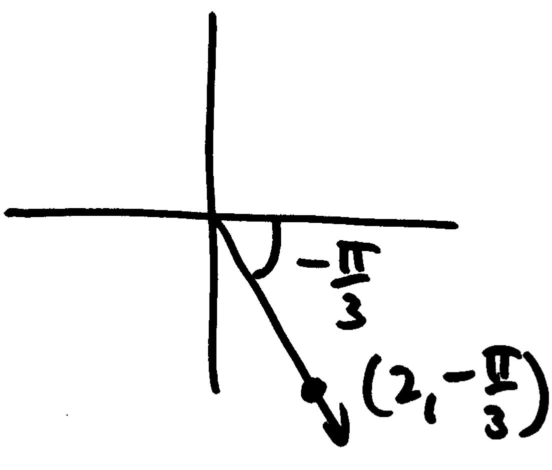


$\left(2, -\frac{2\pi}{3}\right)$



$\left(-2, \frac{2\pi}{3}\right)$



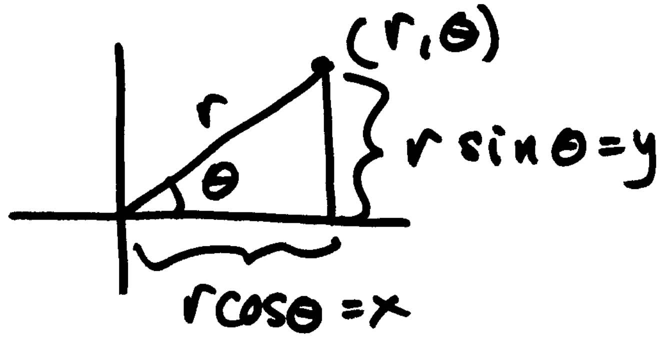


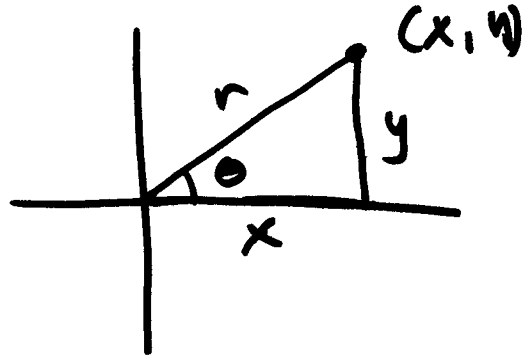
2. Polar $\longleftrightarrow$ Cartesian (Rectangular)

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$r^2 = x^2 + y^2$

$\tan\theta = \dfrac{y}{x}$





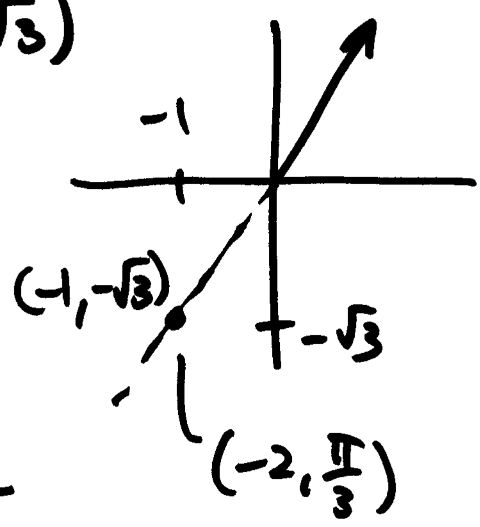
e.g $\left(-2, \frac{\pi}{3}\right) \longrightarrow \left(-1, -\sqrt{3}\right)$

polar

$r = -2 \quad \theta = \dfrac{\pi}{3}$

$x = -2\cos\left(\frac{\pi}{3}\right) = -1$

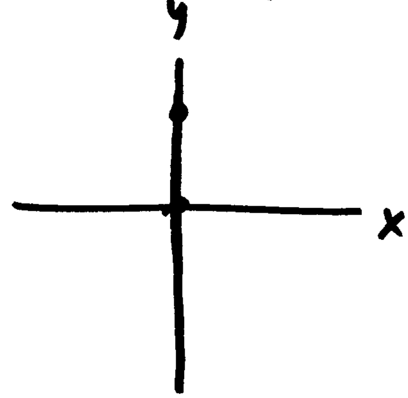
$y = -2\sin\left(\frac{\pi}{3}\right) = -\sqrt{3}$



C. Curves represented by polar equations

#26 $r\cos\theta = 0 \quad \longrightarrow \quad x = 0$

polar eqn. rectangular.



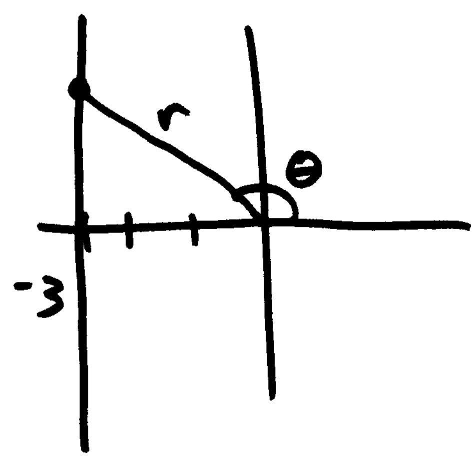
$x=0$

#28 $r = -3\sec\theta$

$r = \dfrac{-3}{\cos\theta}$

$r\cos\theta = -3$

$x = -3$



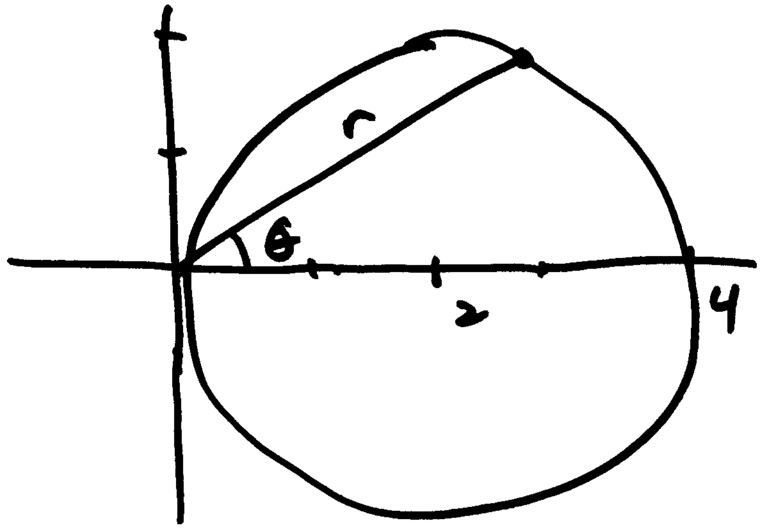
$x=-3$

e.g. $[r = 4\sin\theta]\,r$

$\underbrace{r^2}_{x^2+y^2} = 4\underbrace{r\sin\theta}_{y} \quad \longrightarrow \quad x^2+y^2 = 4y$

$x^2 + (y^2 - 4y + 4) = 4$

$x^2 + (y-2)^2 = 4$ radius 2 ctr $(0,2)$



10.6 Polar graphs.

Idea: Usually look at $y = f(x)$

Now look at $r = f(\theta)$.

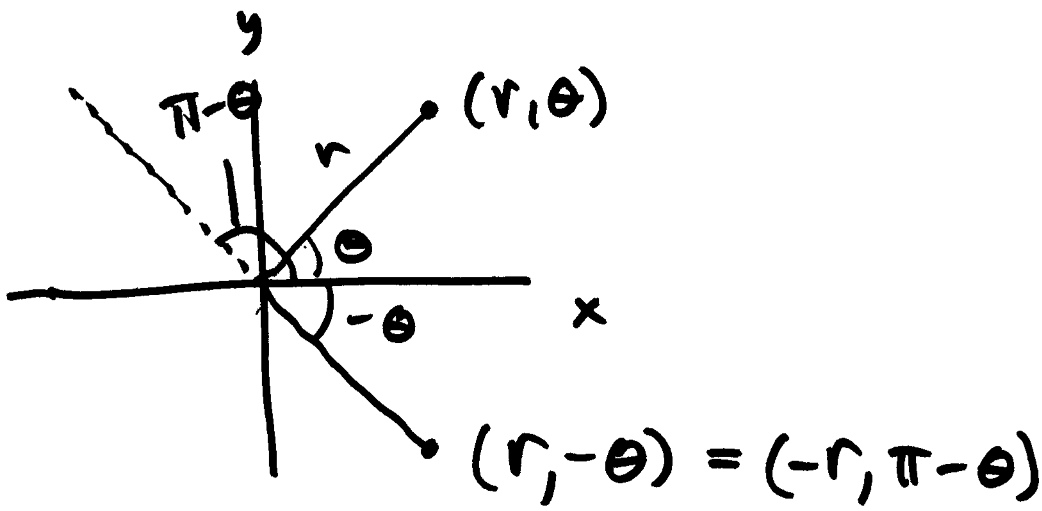
Note: $r = f(\theta)$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

Can treat graph as a set of parametric equations with parameter θ.

Note: We can consider symmetry in these graphs.

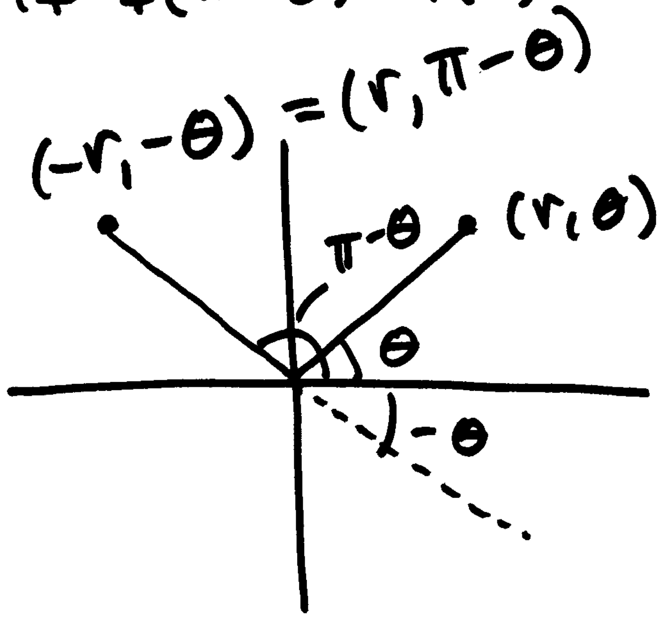


x-axis symmetry if $f(-\theta) = r = f(\theta)$

or if $f(\pi - \theta) = -r = -f(\theta)$

y-axis symmetry if $f(-\theta) = -r = -f(\theta)$

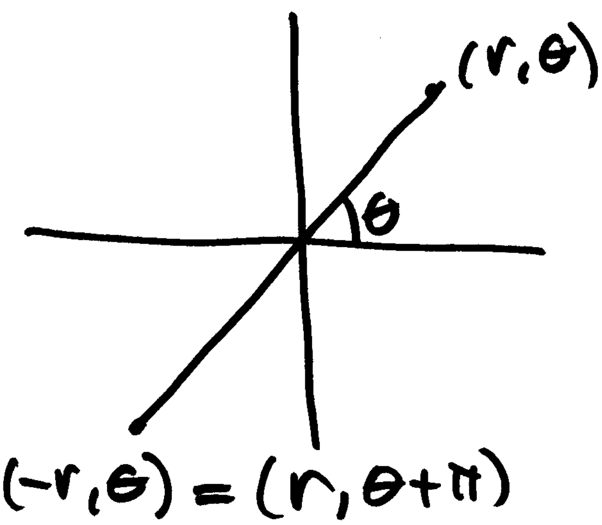
or if $f(\pi - \theta) = f(\theta)$



symmetry about origin.

if $f(\theta) = -f(\theta)$ (NEVER)

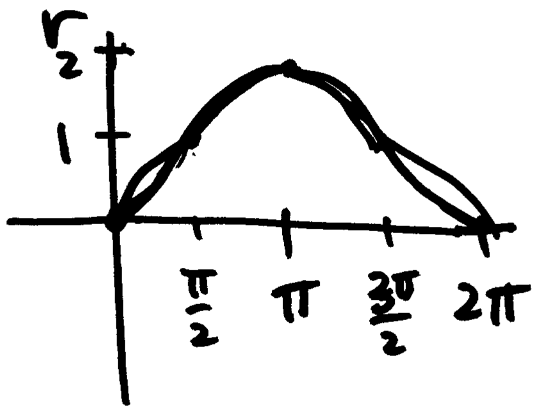
or if $f(\theta + \pi) = f(\theta)$

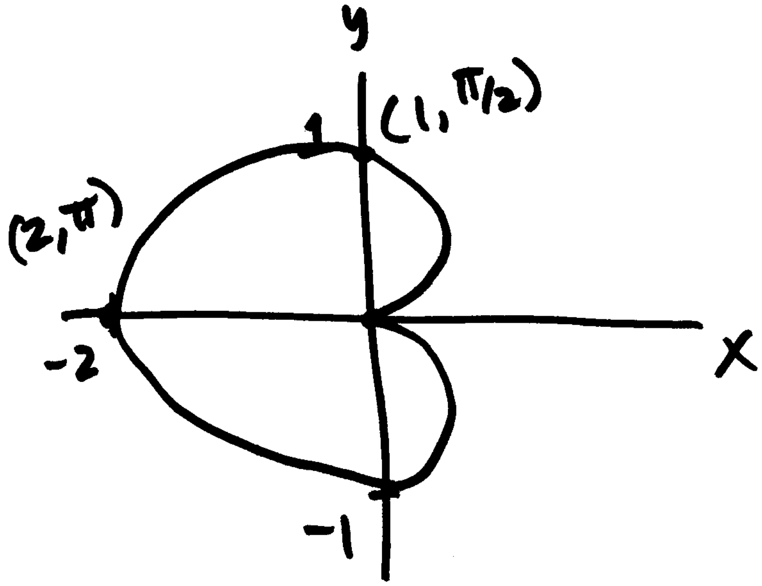


e.g. $r = 1 - \cos(\theta)$ draw graph.

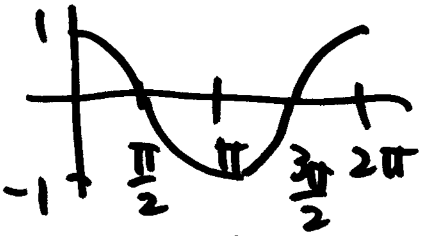
symmetry : x-axis

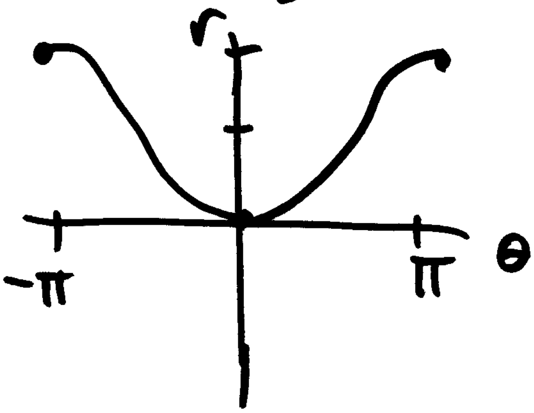
Graph the function as though were in rect. coords.





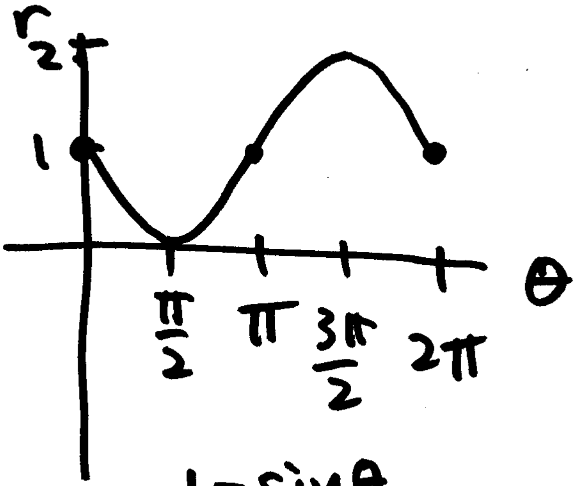
cardioid



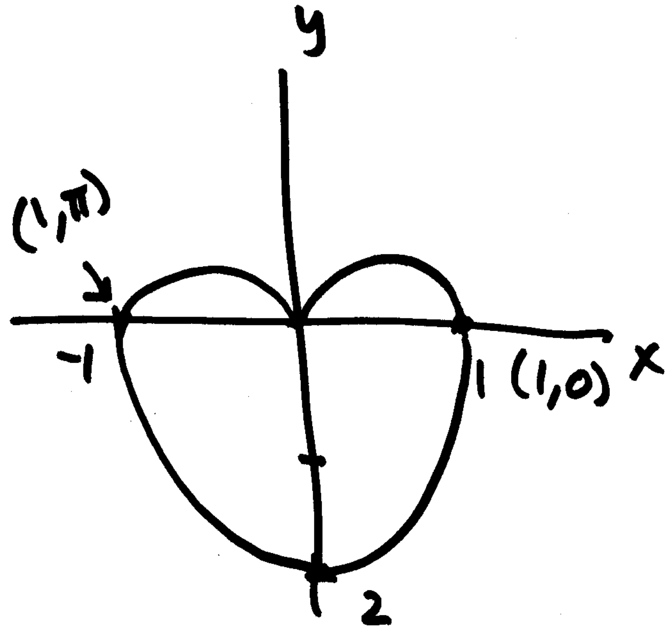


e.g. $r = 1 - \sin\theta$

symmetry: y-axis



$1 - \sin\theta$



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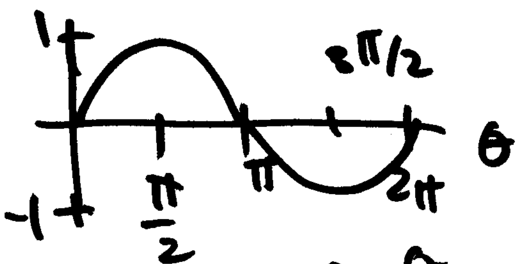


$\sin\theta$

eg $r^2 = \sin\theta$, $r = \pm\sqrt{\sin\theta}$

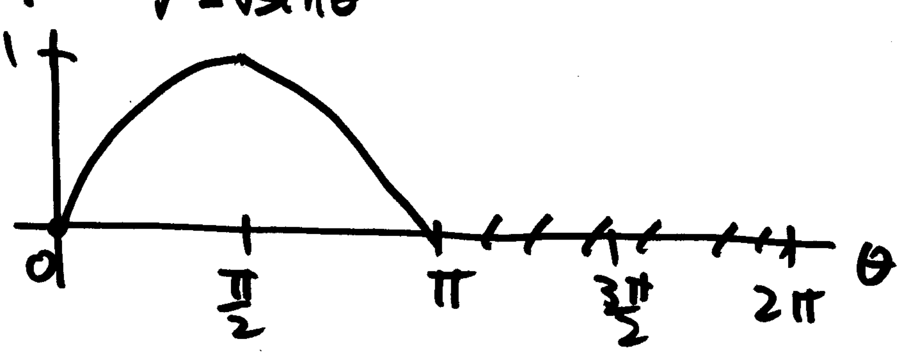
symmetry: x-axis YES

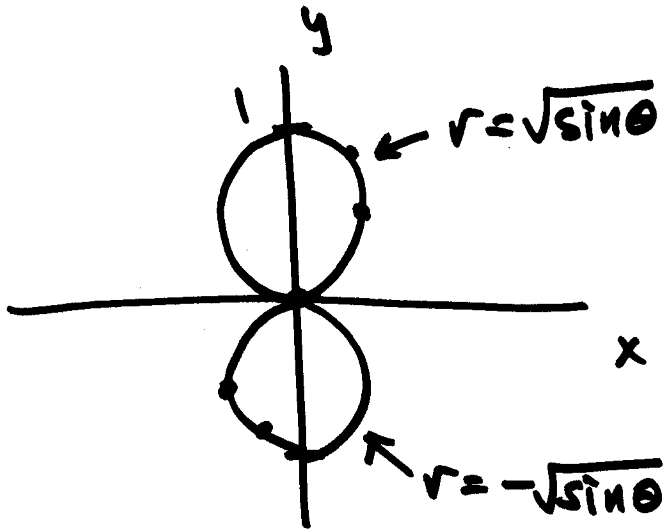
(r,θ) , $(r,-\theta)$ on graph at same time

$(-r, \pi-\theta)$ OK since $\sin(\pi-\theta) = \sin(\theta)$

y-axis YES

$r = \sqrt{\sin\theta}$




y
1
$r = \sqrt{\sin\theta}$
x
$r = -\sqrt{\sin\theta}$