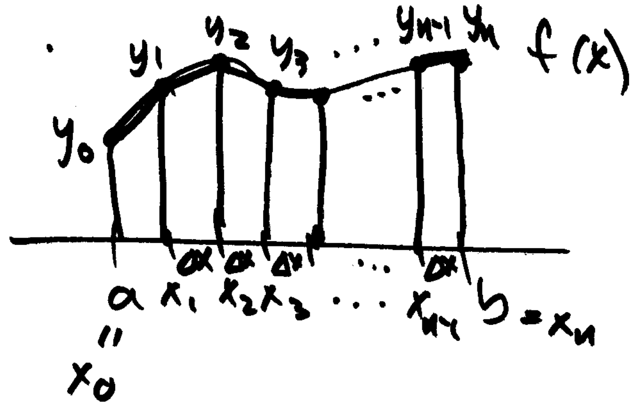
Numerical Integration (contd).

1. Trapezoid rule



$\Delta x = \dfrac{b-a}{n}$

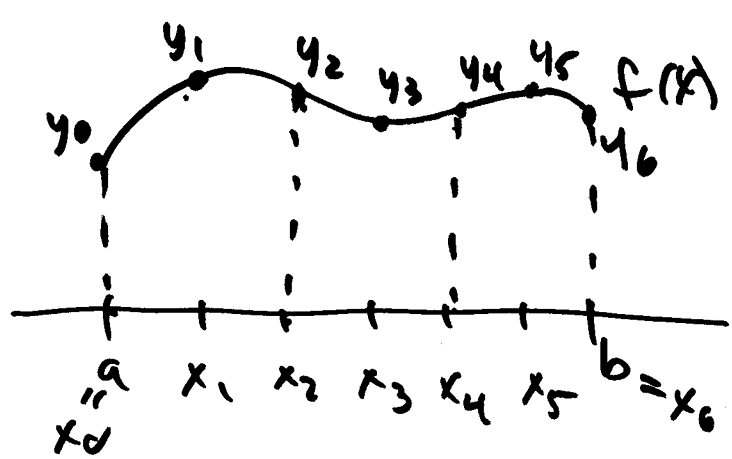
$f(x_i) = y_i$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

$$|E_T| \leq \frac{M(b-a)^3}{12n^2} \quad \text{where } |f''(x)| \leq M \text{ for } a \leq x \leq b$$

Main point: Error decreases like $\frac{1}{n^2}$ so if you double n, you reduce error by $\frac{1}{4}$.

2. Simpson's Rule



Find the quadratic polynomial (i.e. the parabola) interpolating each triple of points.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+4y_{n-1}+y_n\right)$$

$$|E_S| \le \frac{M(b-a)^5}{180n^4} \quad \text{where } |f^{(4)}(x)| \le M \text{ for } a \le x \le b.$$

$\boxed{\text{Note: } n \text{ must be even}}$

Main point: Error drops like $\frac{1}{n^4}$ so if you double n, then error drops ~~like~~ by $\frac{1}{16}$.

$$\operatorname{erf}(x) = \int_0^x e^{-\pi t^2}\,dt$$

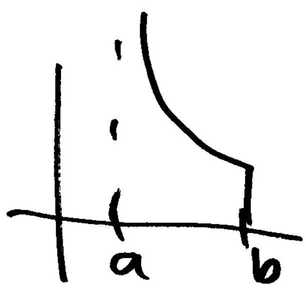
$$\int \frac{4}{x} + \frac{-x+1}{x^2+1}\,dx = \int \frac{4}{x}\,dx + \int \frac{-x+1}{x^2+1}\,dx$$

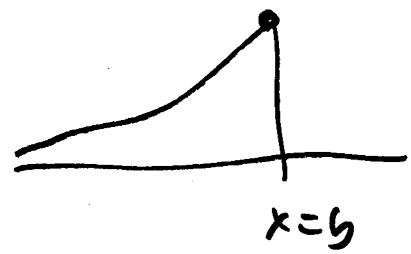
$$= \int \frac{4}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= 4\ln x - \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + c$$

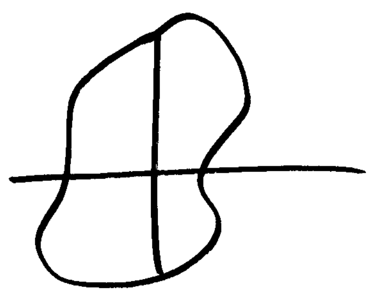
8.8 Improper Integrals.

$\int_a^b f(x)\,dx$, $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$

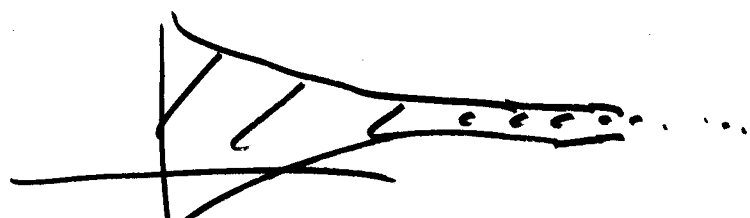




Idea: When does an unbounded region have finite area?

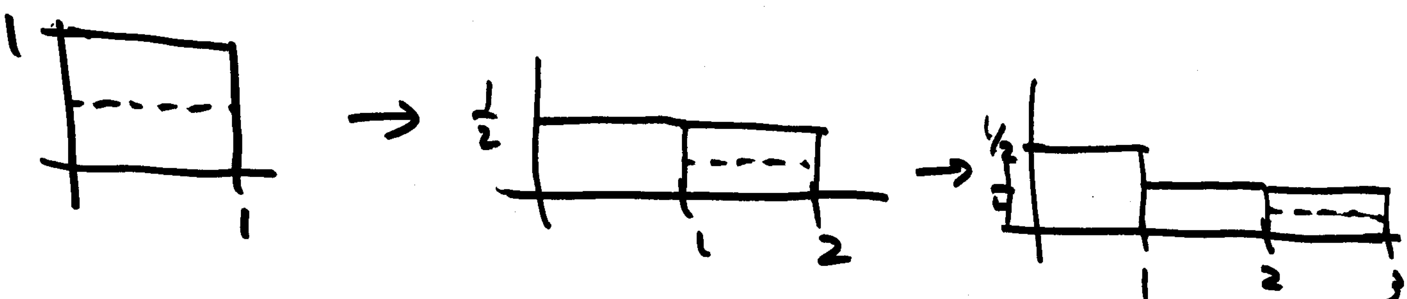


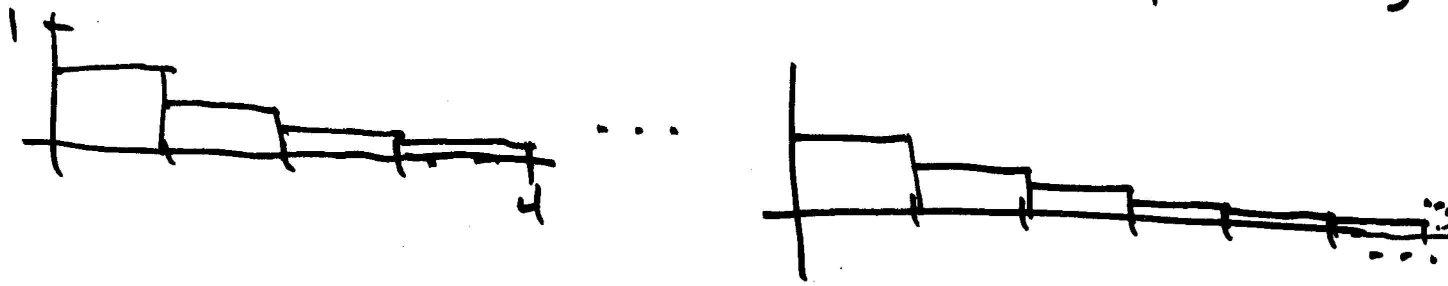
bounded.

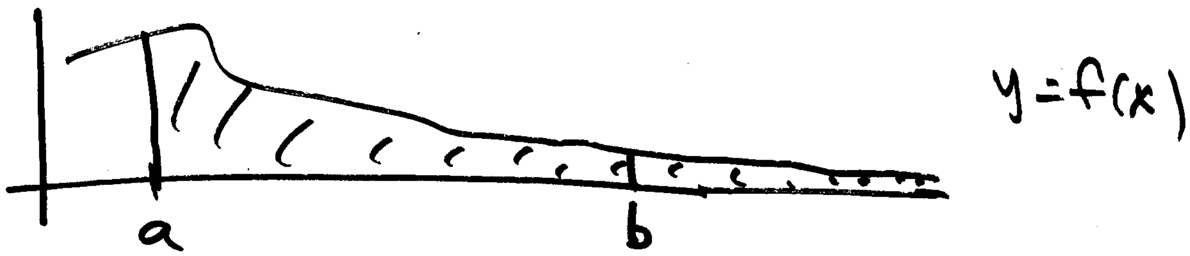


unbounded.

Example: Infinite extent/finite area







Looking at: $\int_a^\infty f(x)\,dx$

Definition: $\lim\limits_{b\to\infty}\int_a^b f(x)\,dx = \int_a^\infty f(x)\,dx$

<u>e.g.</u> $\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim\limits_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx$

$\int_1^b \frac{\ln(x)}{x^2}\,dx$ Parts: $u=\ln(x)$ $dv = x^{-2}dx$

$du = \frac{1}{x}dx$ $v = -x^{-1} = -\frac{1}{x}$

$= -\frac{1}{x}\ln(x)\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx$

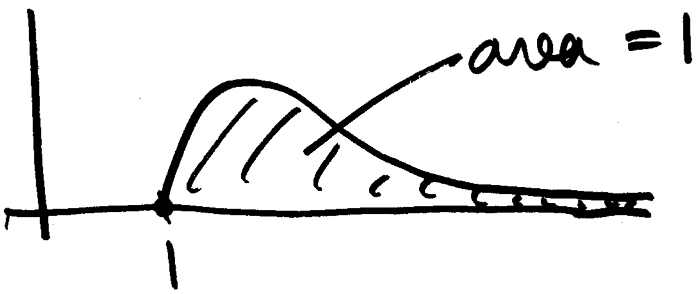
$= -\frac{1}{b}\ln(b) + 1\cdot \underbrace{\cancel{\ln(1)}}_{0} + \left(-\frac{1}{x}\Big|_1^b\right)$

$= -\frac{1}{b}\ln(b) - \frac{1}{b} + 1$

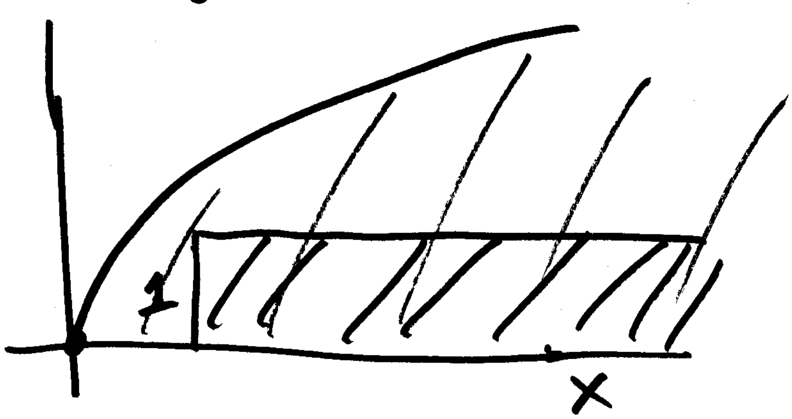
$= \lim\limits_{b\to\infty}\left(-\frac{1}{b}\ln(b) - \underbrace{\cancel{\frac{1}{b}}}_{0} + 1\right) = 1 - \lim\limits_{b\to\infty}\frac{\ln(b)}{b}$

$$\overset{L'H}{=} 1 - \lim_{b\to\infty} \frac{\frac{1}{b}}{1} = 1 - \lim_{b\to\infty} \overset{0}{\cancel{\tfrac{1}{b}}} = 1.$$

$$\therefore \int_1^\infty \frac{\ln(x)}{x^2}\,dx = 1$$



<u>eg.</u> $\int_0^\infty \sqrt{x}\,dx = \infty$ (intuitively)

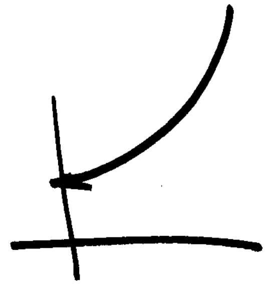


$$\lim_{b\to\infty} \int_0^b x^{1/2}\,dx = \lim_{b\to\infty} \frac{2}{3} x^{3/2} \Big|_0^b$$

$$= \lim_{b\to\infty} \frac{2}{3} b^{3/2} = \infty.$$

Rates of growth.

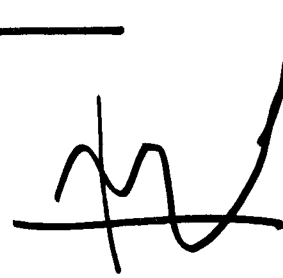
1. exponential functions



$f(x) = e^x$ $\quad f(x) = 2^x$ $\quad f(x) = e^{5x}$

$f(x) = 2^{x^2}$ $\quad f(x) = a^x$ if $a > 1$ <u>and</u> $x \to \infty$ then $a^x \to \infty$.

2. ~~polynomial~~ power functions



$f(x) = x^2$ $\quad f(x) = x^3 + x^{2/3} + 2x$

LOOK LIKE $f(x) = x^{\alpha}$ $\alpha > 0$

For such functions $\lim_{x \to \infty} f(x) = \infty$.

$f(x) = \sqrt{x^2+1}$

3. logarithmic functions

$f(x) = \ln(x)$ $\quad f(x) = \log_3(x)$

---

exponential growth

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$

polynomial growth

polynomial

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

exponential

$$\lim_{x\to\infty} \frac{e^x}{x^3} \quad \left(\text{Form } \frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \begin{matrix} \leftarrow \text{polynomial} \\ = \infty \\ \leftarrow \text{logarithmic} \end{matrix}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \begin{matrix} \leftarrow \text{logarithmic} \\ \leftarrow \text{polynomial} \end{matrix} = 0$$

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} \left(\text{Form } \frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2x}$$

$$= \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

---

e.g. $\displaystyle \lim_{x\to\infty} \frac{e^x}{2^x} = \lim_{x\to\infty} \left(\frac{e}{2}\right)^x = \infty.$

$\qquad > 1$

$$\lim_{x\to\infty} \frac{2x+3}{x^2+1} = \lim_{x\to\infty} \frac{2x}{x^2} = \lim_{x\to\infty} \frac{2}{x} = 0$$

$$\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2}} = \lim_{x\to\infty} \frac{2x}{x} = 2.$$

---

e.g. $$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx$$

$$\int_1^b \frac{\ln(x)}{x}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$x=1 \quad u=0$

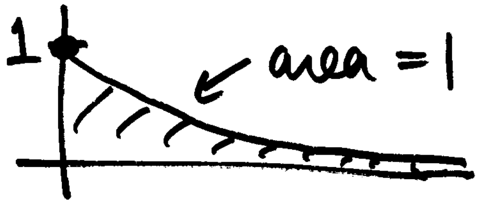
$x=b \quad u=\ln(b)$

$$= \int_0^{\ln(b)} u\,du = \frac{1}{2}u^2\Big|_0^{\ln(b)} = \frac{1}{2}\ln(b)^2$$

$$= \lim_{b\to\infty} \frac{1}{2}\ln(b)^2 = \infty.$$

area $= \infty$.

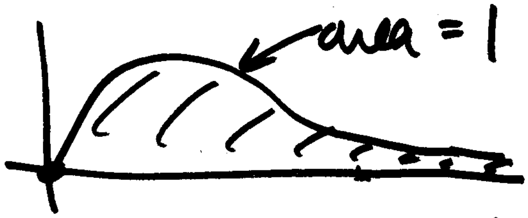
e.g. $\displaystyle\int_0^\infty e^{-x}\,dx$



$$= \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x} \Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[-e^{-b} + e^{0} \right] = \lim_{b\to\infty} \left[1 - e^{-b} \right] = 1$$

e.g. $\displaystyle\int_0^\infty x e^{-x}\,dx$



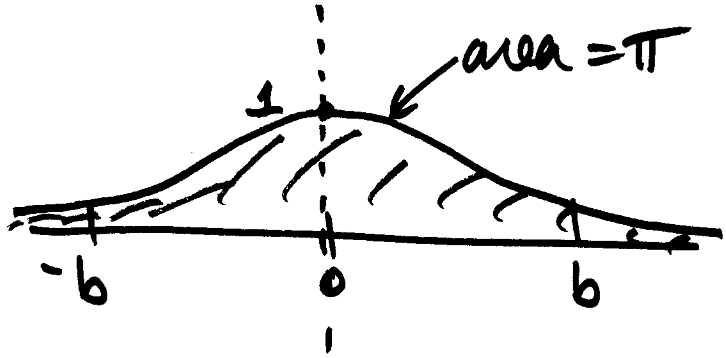
$$= \lim_{b\to\infty} \int_0^b x e^{-x}\,dx = \lim_{b\to\infty} -\underbrace{b e^{-b}}_{\frac{b}{e^b}}{}^{\to 0} + 1 - e^{-b}{}^{\to 0} = 1$$

$$\int_0^b x e^{-x}\,dx = \begin{bmatrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{bmatrix}$$

$$= -xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx$$

$$= -be^{-b} + \int_0^b e^{-x}\,dx = -be^{-b} + 1 - e^{-b}$$

e.g. $$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$



$$= \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

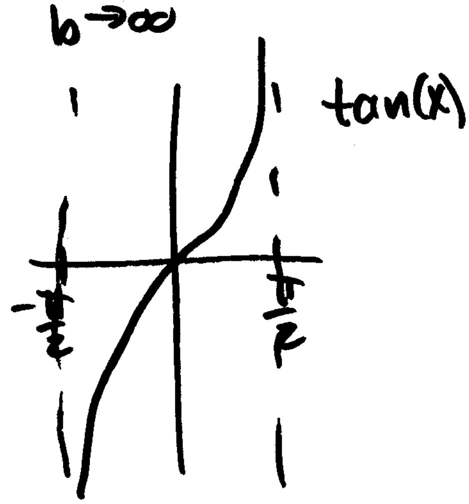
$$= \lim_{b\to\infty} \int_{-b}^{0} \frac{1}{x^2+1}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+1}\,dx$$

$$= \underbrace{\lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+1}\,dx}_{\text{by symmetry}} + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+1}\,dx$$

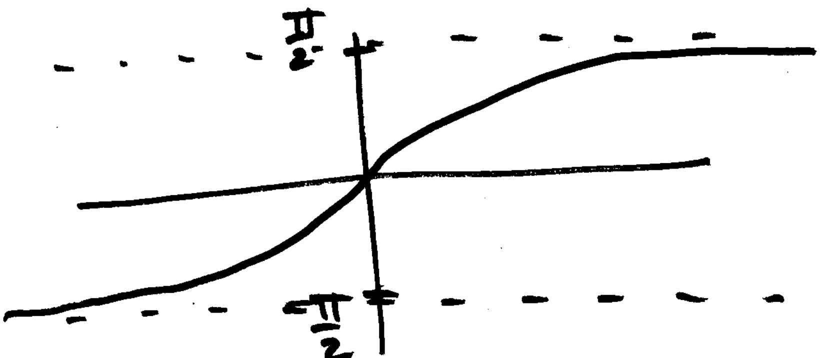
$$= \lim_{b\to\infty} 2 \int_{0}^{b} \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} 2\cdot \tan^{-1}(x)\Big|_{0}^{b} = \lim_{b\to\infty} 2\cdot\left(\tan^{-1}(b) - \overset{0}{\cancel{\tan^{-1}(0)}}\right)$$

$$= 2 \lim_{b\to\infty} \tan^{-1}(b).$$



$$= 2\cdot\frac{\pi}{2} = \pi \,//$$

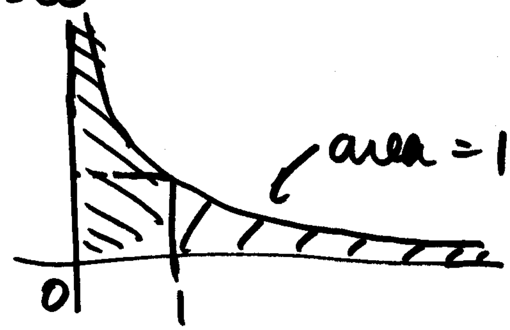


$\tan^{-1}(x)$.

p-integrals : $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx \qquad p>0$

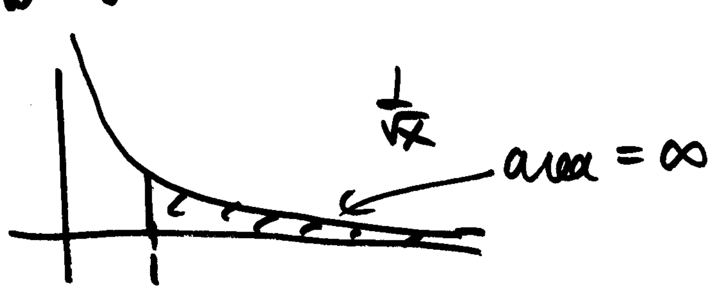
e.g. (p=2) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$

$\displaystyle = \lim_{b\to\infty}\left[-\frac{1}{x}\Big|_1^b\right] = \lim_{b\to\infty}\left(-\overset{\to 0}{\cancel{\tfrac{1}{b}}}+1\right) = 1$



e.g. (p=½) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx$

$\displaystyle = \lim_{b\to\infty}\left[2x^{1/2}\Big|_1^b\right] = \lim_{b\to\infty}\left(2\sqrt{b}-2\right) = \infty.$



$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx \qquad p \neq 1$$

$$= \lim_{b\to\infty} \left[\frac{1}{-p+1} x^{-p+1} \Big|_1^b \right] = \lim_{b\to\infty} \left(\frac{1}{1-p} b^{1-p} - \frac{1}{1-p} \right)$$

If $p > 1$, $1-p < 0$ and $\lim_{b\to\infty} b^{1-p} = 0$

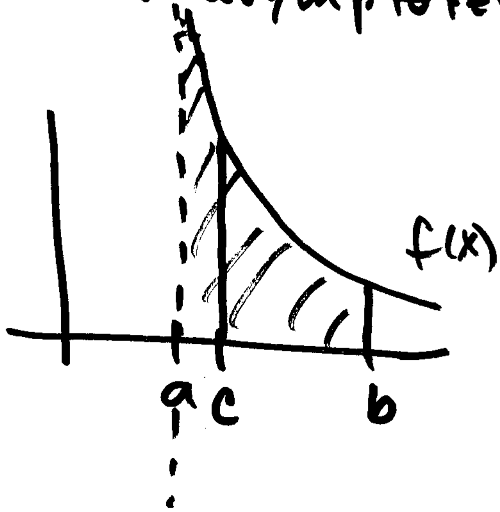
If $p < 1$, $1-p > 0$ and $\lim_{b\to\infty} b^{1-p} = \infty$

$$\boxed{\begin{array}{l} \text{If } p > 1 \text{ then } \displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} < \infty \\ \text{If } p \leq 1 \text{ then } \displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \infty. \end{array}}$$

If $p = 1$: $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

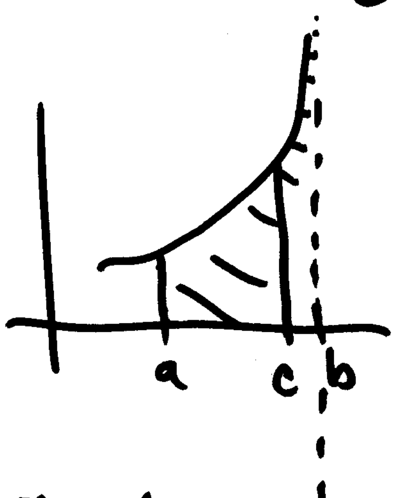
$$= \lim_{b\to\infty} \left[\ln(x) \Big|_1^b \right] = \lim_{b\to\infty} \ln(b) = \infty.$$

2. Vertical asymptotes.



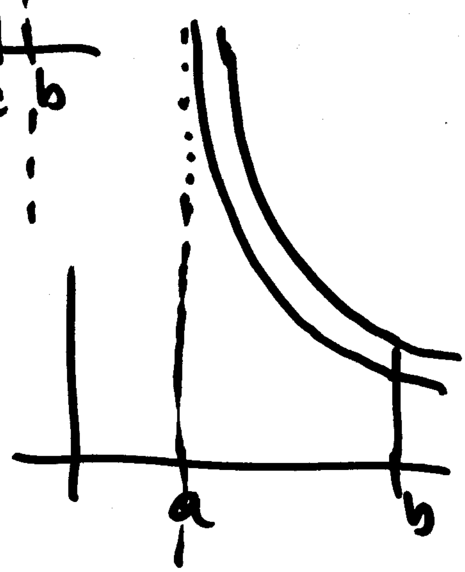
Infinite region.
Finite area?

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

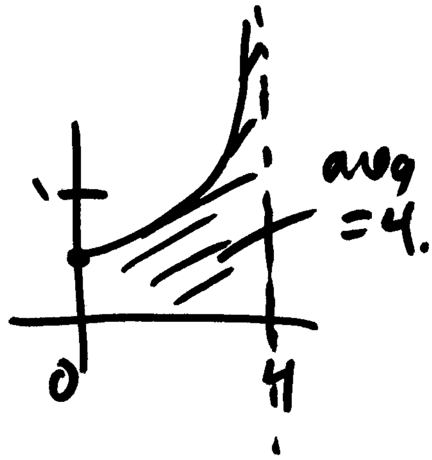


$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

Intuitively



e.g. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^b \frac{dx}{\sqrt{4-x}}$



$\displaystyle -\int_0^b (4-x)^{-1/2}(-dx)$

$u = 4-x$
$du = -dx$

$x=0 \quad u=4$
$x=b \quad u=4-b$

$\displaystyle = -\int_4^{4-b} u^{-1/2}\,du = \int_{4-b}^{4} u^{-1/2}\,du$

$\displaystyle = 2u^{1/2}\Big|_{4-b}^{4} = 4-2(4-b)^{1/2}$

$\displaystyle = \lim_{b\to 4^-} 4 - 2(\overset{\to 0}{4-b})^{1/2} = 4$ //

e.g. $\displaystyle\int_0^1 \frac{1}{t^2-1}\,dt = \lim_{b\to 1^-} \int_0^b \frac{1}{t^2-1}\,dt$

$\displaystyle\int_0^b \frac{dt}{t^2-1}$

$\displaystyle\frac{1}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1}$

$1 = A(t-1) + B(t+1)$ $\quad t=-1$

$1 = -2A$ $\quad \boxed{A = -\frac{1}{2}}$ $\quad t=1$ $\quad \boxed{B = \frac{1}{2}}$

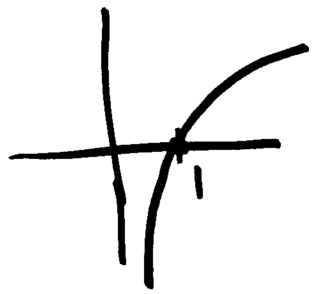
$$= -\frac{1}{2}\int_0^b \frac{dt}{t+1} + \frac{1}{2}\int_0^b \frac{dt}{t-1}$$

$$= -\frac{1}{2}\ln(|t+1|) + \frac{1}{2}\ln(|t-1|)\Big|_0^b$$

$$= -\frac{1}{2}\ln(b+1) + \frac{1}{2}\ln|b-1| + \frac{1}{2}\cancelto{0}{\ln(1)} - \frac{1}{2}\cancelto{0}{\ln|-1|}$$

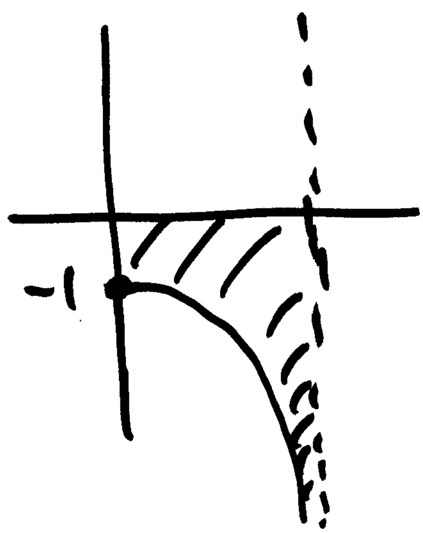
$$= -\frac{1}{2}\ln(b+1) + \frac{1}{2}\ln|b-1|$$

$$= \lim_{b\to 1^-}\left(-\frac{1}{2}\ln(b+1) + \underbrace{\frac{1}{2}\ln|b-1|}_{\frac{1}{2}\ln(1-b)}\right)$$



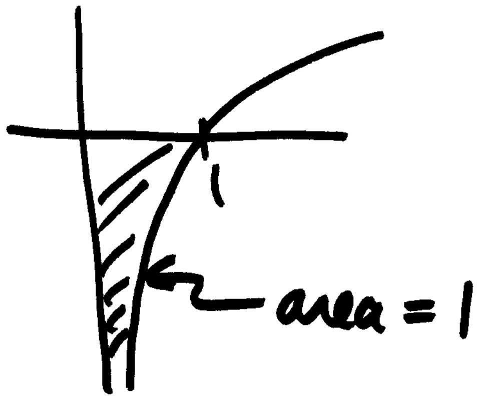
$$= -\frac{1}{2}\ln(2) - \infty$$

$$= -\infty \quad //$$



$$\frac{1}{t^2-1}$$

e.g. $\int_0^1 \ln(x)\,dx$



$$= \lim_{b\to 0^+} \int_b^1 \ln(x)\,dx$$

$$\int_b^1 \ln(x)\,dx \qquad u=\ln(x) \quad dv=dx$$
$$du=\frac{1}{x}\,dx \quad v=x$$

$$= x\ln(x)\Big|_b^1 - \int_b^1 x\cdot\frac{1}{x}\,dx$$

$$= 1\cdot \overset{0}{\cancel{\ln(1)}} - b\cdot\ln(b) - (1-b)$$

$$= b-1-b\ln(b)$$

$$= \lim_{b\to 0^+}\left(\underbrace{b}_{0}-1-\underbrace{b\ln(b)}_{0}\right) = -1$$

$$\lim_{b\to 0^+} b\ln(b) \quad \text{Form } (0\cdot\infty)$$

$$= \lim_{b\to 0^+} \frac{\ln(b)}{\frac{1}{b}} \quad \text{Form}\left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{b\to 0^+} \frac{\frac{1}{b}}{-\frac{1}{b^2}} = \lim_{b\to 0^+} -b = 0$$

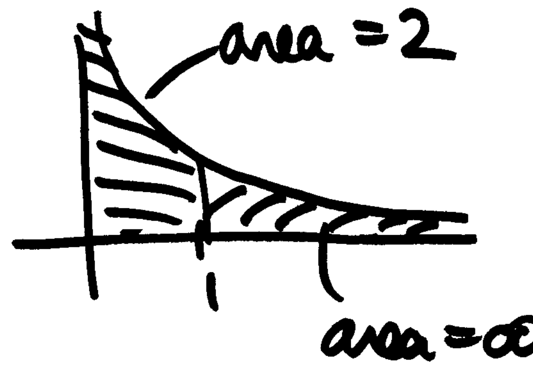
p-integrals, $p>0$

$$\int_0^1 \frac{1}{x^p}\,dx$$

e.g. $\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+} \int_b^1 \frac{1}{x^2}\,dx$

$$= \lim_{b\to 0^+}\left[-\frac{1}{x}\Big|_b^1\right] = \lim_{b\to 0^+}\left(-1+\frac{1}{b}\right) = \infty$$

e.g. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$



$$= \lim_{b\to 0^+} \int_b^1 x^{-1/2}\,dx$$

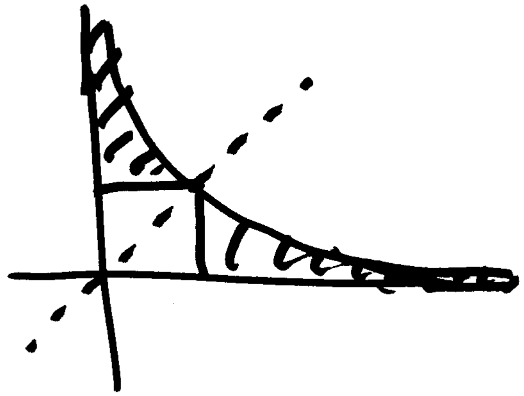
$$= \lim_{b\to 0^+} 2x^{1/2}\Big|_b^1 = \lim_{b\to 0^+}\left(2-2b^{1/2}\right) = 2$$

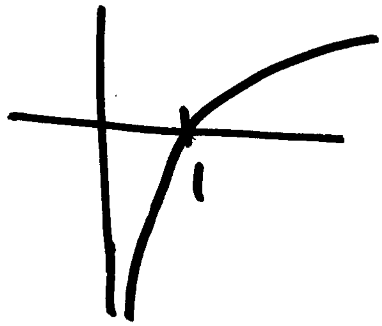
In fact: If $p \geq 1$ then $\int_0^1 \frac{1}{x^p}\,dx = \infty$

If $p<1$ then $\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} < \infty$

e.g. $\int_0^1 \frac{1}{x}\,dx = \lim_{b\to 0^+} \int_b^1 \frac{1}{x}\,dx$

$= \lim_{b\to 0^+} \ln(x)\Big|_b^1 = \lim_{b\to 0^+} \left(-\ln(b)\right) = \infty$





3. Convergence Tests.

Idea: Can determine whether an improper integral converges without calculating it (sometimes).

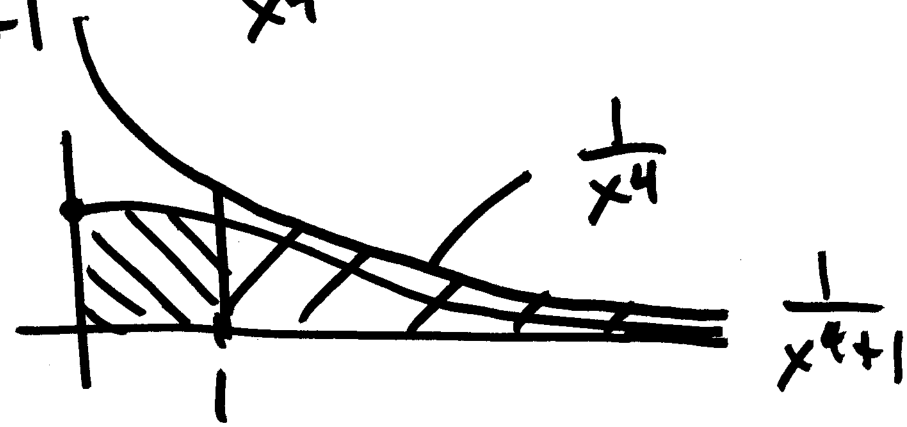
e.g. $\int_0^\infty \frac{dx}{x^4+1}$ Finite or infinite?

$\frac{1}{x^4+1}$

<u>Think</u>: $\frac{1}{x^4+1}$ behaves like $\frac{1}{x^4}$ (if x large)

so we expect $\int_0^\infty \frac{1}{x^4+1}dx$ to behave like $\int_1^\infty \frac{1}{x^4}dx$ which is <u>finite</u> $(p=4>1)$

<u>Proof</u>: $\frac{1}{x^4+1} \leq \frac{1}{x^4}$ for all x



So $\int_1^\infty \frac{1}{x^4+1}dx \leq \int_1^\infty \frac{1}{x^4}dx = \frac{1}{3}$

$\therefore \int_0^\infty \frac{1}{x^4+1}dx < \infty$