MAPLE #2 ONLINE

8.6 Integral Tables.

#4) $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/2}} =$

Looks like #7 with $n=-3/2$, $a=2$, $b=3$

$$= \frac{(2x+3)^{-1/2}}{4}\left[\frac{2x+3}{\frac{1}{2}} - \frac{3}{-\frac{1}{2}}\right] + C$$

$$= \frac{1}{4(2x+3)^{1/2}}\cdot\left[2(2x+3)+6\right] + C$$

$$= \frac{4x+12}{4(2x+3)^{1/2}} + C = \frac{x+3}{(2x+3)^{1/2}} + C$$

Doable: $\displaystyle\frac{1}{2}\int \frac{x\,2dx}{(2x+3)^{3/2}}$ $\quad u=2x+3 \quad x=\frac{1}{2}(u-3)$, $du=2dx$

$$= \frac{1}{2}\int \frac{\frac{1}{2}(u-3)\,du}{u^{3/2}} = \frac{1}{4}\int u^{-1/2} - 3u^{-3/2}\,du$$

$$= \frac{1}{4}\left(2u^{1/2} - 3(-2)u^{-1/2}\right) + C$$

$$= \frac{1}{2}\left(u^{1/2} + 3u^{-1/2}\right) + C = \frac{1}{2}\left((2x+3)^{1/2} + 3(2x+3)^{-1/2}\right) + C$$

$$= \frac{1}{2}\left[(2x+3)^{1/2} + \frac{3}{(2x+3)^{1/2}}\right] + C$$

$$= \frac{1}{2}\left[\frac{2x+3+3}{(2x+3)^{1/2}}\right] + C$$

$$= \frac{1}{2}\left[\frac{2x+6}{(2x+3)^{1/2}}\right] + C = \boxed{\frac{x+3}{(2x+3)^{1/2}} + C} \quad //$$

<u>e.g.</u> #10) $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx = \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C$

Looks like #52 with $a = \frac{1}{2}$

$$= \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C \quad //$$

$$\int \frac{\sqrt{x-x^2}}{x}\,dx$$

$$= \int \frac{\sqrt{1-(x-\frac{1}{2})^2}}{x}\,dx$$

$$x - x^2 = -(x^2 - x + \tfrac{1}{4}) + \tfrac{1}{4}$$

$$= \tfrac{1}{4} - (x-\tfrac{1}{2})^2$$

$$x - \tfrac{1}{2} = \sin\theta$$

$$x = \tfrac{1}{2} + \sin\theta$$

maybe...

#16) $\int q^2\sqrt{25-q^2}\,dq = \frac{625}{8}\sin^{-1}\left(\frac{q}{5}\right)$

$\boxed{\#30:\ a=5}$ $\quad -\frac{1}{8}q\sqrt{25-q^2}\,(25-2q^2)+C$

#44) $\int \frac{\sqrt{2-x}}{\sqrt{x}}\,dx$ $\quad u=2-x \quad x=2-u$

$du=-dx$

$= \cancel{-\int \frac{\sqrt{u}}{\sqrt{2-u}}\,du}$ NO HELP

$u=\sqrt{2-x}=(2-x)^{1/2}$

$du=-\frac{1}{2}(2-x)^{-1/2}\,dx$

$=-\frac{1}{2}u^{-1}\,dx=-\frac{1}{2u}\,dx$

$dx=-2u\,du$

$u^2=2-x \quad x=2-u^2$

$= \int \frac{u}{\sqrt{2-u^2}}\cdot -2u\,du$

$= -2\int \frac{u^2}{\sqrt{2-u^2}}\,du$

$\boxed{\#33,\ a=\sqrt{2}}$

$= -2\left[\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)-\frac{1}{2}u\sqrt{2-u^2}\right]+C$

$\int u\cdot\underbrace{\frac{-2u}{\sqrt{2-u^2}}\,du}_{dv}$

resub $u=\sqrt{2-x}$

etc...

maybe parts...

Definitely trig subst.

#38) $\int \cos\left(\frac{\theta}{2}\right)\cos(7\theta)\,d\theta$

#62 c) with $a=\frac{1}{2}$, $b=7$

8.7 Numerical Integration

Idea: Integrals are fundamentally harder than derivatives and some integrals cannot be done in a reasonable form.

e.g. $\int e^{x^2}dx$ $\qquad \frac{e^{x^2}}{2x}$ ← ~~also~~ does not work.

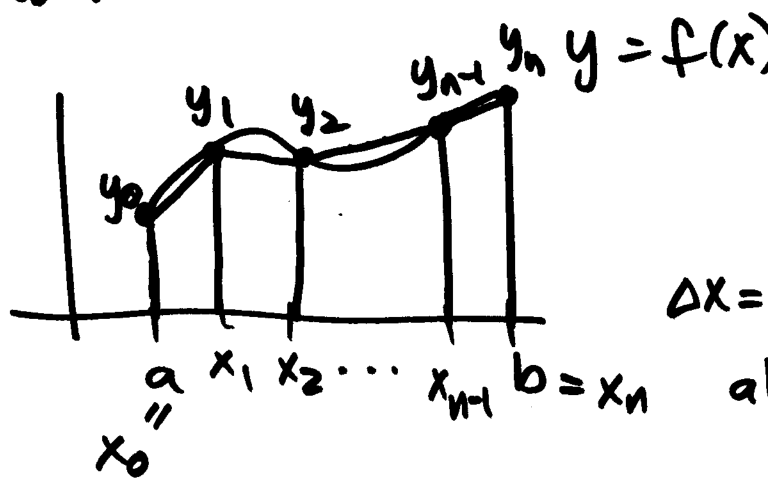
$$\int_0^x e^{-t^2}\,dt = \mathrm{erf}(x)$$

e.g. Also $\int e^{-x^2}dx$ cannot be done

So the only way to work with them is to approximate numerically.

1. Trapezoid rule.

Idea:



$\Delta x = x_{i+1} - x_i$ all the same

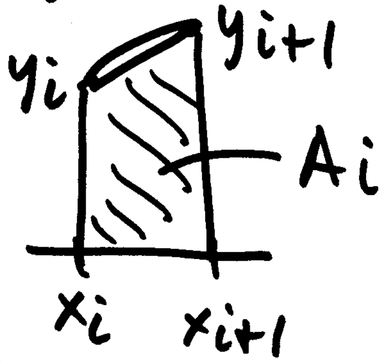
Want $\int_a^b f(x)\,dx$

① Divide $[a,b]$ into n equal subintervals

$$\Delta x = \frac{b-a}{n}$$

② Look at $y_i = f(x_i)$ $\quad i = 0, 1, 2, \dots, n$

③ Approximate area under graph for x in $[x_i, x_{i+1}]$ by area of a trapezoid:



$$A_i = \left(\frac{y_{i+1} + y_i}{2}\right)(x_{i+1} - x_i)$$

$$= \left(\tfrac{1}{2} y_{i+1} + \tfrac{1}{2} y_i\right)\Delta x$$

④ $$\int_a^b f(x)\,dx \approx \left(\tfrac{1}{2} y_0 + \tfrac{1}{2} y_1\right)\Delta x + \left(\tfrac{1}{2} y_1 + \tfrac{1}{2} y_2\right)\Delta x + \left(\tfrac{1}{2} y_2 + \tfrac{1}{2} y_3\right)\Delta x + \cdots + \left(\tfrac{1}{2} y_{n-1} + \tfrac{1}{2} y_n\right)\Delta x$$

$$= \Delta x\left(\tfrac{1}{2} y_0 + y_1 + y_2 + y_3 + \cdots + y_{n-1} + \tfrac{1}{2} y_n\right)$$

$$= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

<u>e.g.</u> $\displaystyle\int_0^1 (t^3+t)\,dt \qquad n=4$

Trapezoid rule:

$\Delta x = \frac{1}{4}$

$$\begin{array}{ccccc} 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$

$y_0 = x_0^3 + x_0 = 0$

$y_1 = x_1^3 + x_1 = \left(\frac{1}{4}\right)^3 + \frac{1}{4} = \frac{1}{64} + \frac{1}{4} = \frac{17}{64}$

$y_2 = x_2^3 + x_2 = \left(\frac{1}{2}\right)^3 + \frac{1}{2} = \frac{5}{8}$

$y_3 = x_3^3 + x_3 = \left(\frac{3}{4}\right)^3 + \frac{3}{4} = \frac{27}{64} + \frac{3}{4} = \frac{75}{64}$

$y_4 = x_4^3 + x_4 = (1)^3 + 1 = 2$

$$\int_0^1 (t^3+t)\,dt \approx \frac{1}{8}\left(0 + \frac{17}{32} + \frac{5}{4} + \frac{75}{32} + 2\right)$$

$$= \frac{1}{8}\left(\frac{17}{32} + \frac{40}{32} + \frac{75}{32} + \frac{64}{32}\right) = \frac{1}{8}\cdot\frac{196}{32}$$

$$= .765625$$

Compare to exact answer:

$$\int_0^1 (t^3+t)\,dt = \frac{1}{4}t^4 + \frac{1}{2}t^2 \Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} = .75$$

$$\text{error} = |.765625 - .75| = .015625$$

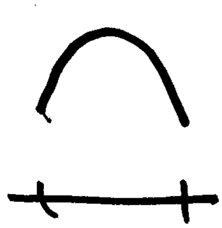
<u>Error</u> <u>estimate</u>.

If f'' cont on $[a,b]$ and if

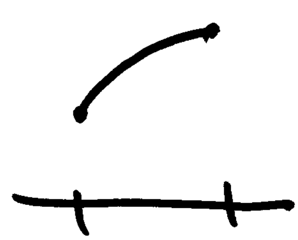
$|f''(x)| \le M$ for all x in $[a,b]$ then

$$|E_T| \le \frac{M(b-a)^3}{12n^2}$$

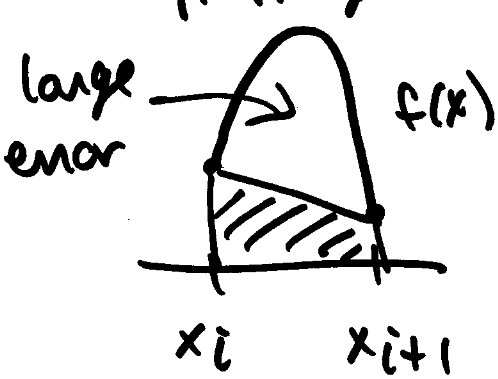
a. f'' measures curvature of graph of f

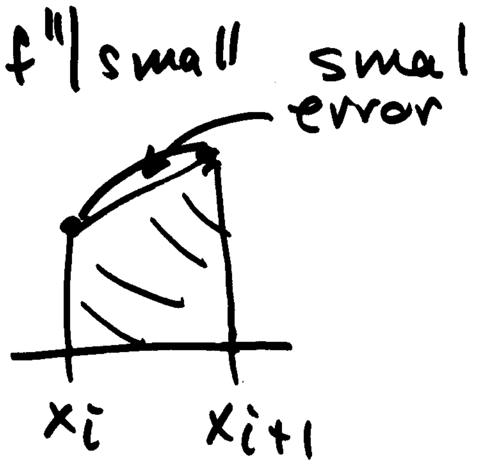


$|f''|$ large



$|f''|$ small





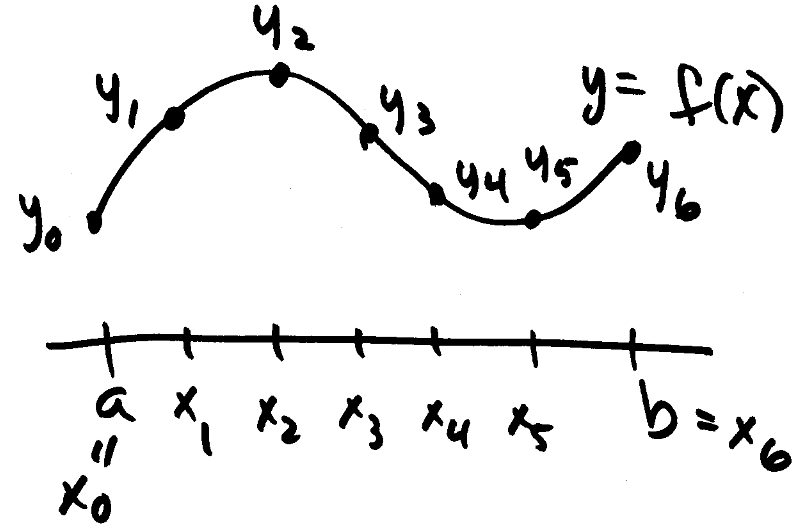
b. Clearly E_T decreases as n increases.

In fact error decreases like $\frac{1}{n^2}$

This means that if you double n you tend to decrease E_T by factor of 4

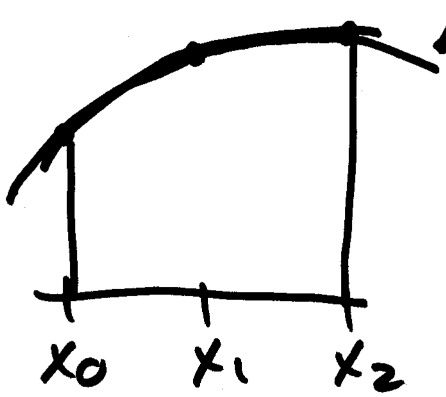
2. Simpson's Rule

Idea:



Take points 3 at a time

Interpolate a quadratic function through the points.

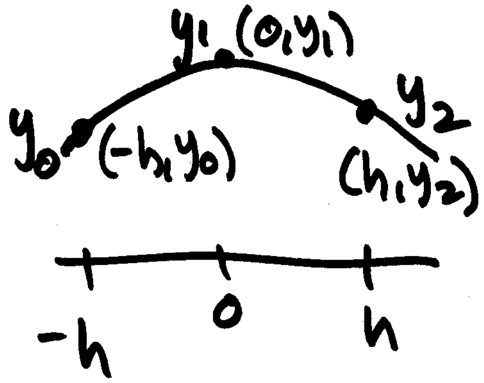


On interval $[x_0, x_2]$ approximate

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} (\text{quadratic})\,dx$$

Derivation:

<u>Idea</u>: For convenience arrange points as:



Spse $Ax^2 + Bx + C$ <u>interpolates</u> points

$$\int_{-h}^{h} Ax^2 + Bx + C \, dx = \frac{2}{3} A h^3 + 2Ch$$

What are A, B, C?

$A(-h)^2 + B(-h) + C = y_0 \qquad Ah^2 - Bh + C = y_0$

$A(0)^2 + B(0) + C = y_1 \qquad \boxed{C = y_1}$

$A(h)^2 + B(h) + C = y_2 \qquad Ah^2 + Bh + C = y_2$

$2Ah^2 + 2C = y_0 + y_2$

$$= \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(\underbrace{2Ah^2 + 2C}_{y_0 + y_2} + \underbrace{4C}_{4y_1})$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Back to full integral

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) + \frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$$
$$+ \frac{\Delta x}{3}(y_4 + 4y_5 + y_6)$$

$$= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

<u>Note</u>: For this to work n must be <u>even</u>

More later...