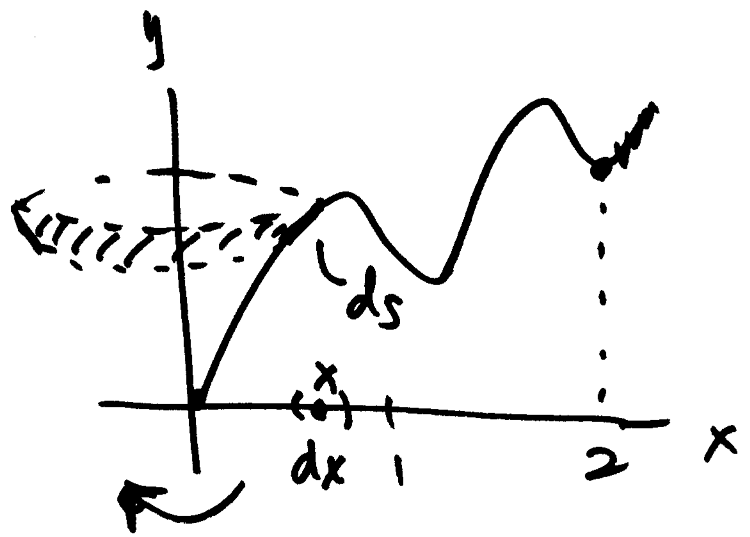
Exam 2 Friday 6.5, 7.2, 7.4, 8.1–8.4

Review MAPLE 1: $y = f(x) = x + \sin^2(\pi x)$

$0 \leq x \leq 2$



Can't solve for x in terms of y.

$$\boxed{\begin{aligned} x &= t \\ y &= t + \sin^2(\pi t) \end{aligned}}$$

x-integral: $dS = 2\pi(\text{radius})(\text{slant height})$

$= 2\pi x \, ds$

$ds = (dx^2 + dy^2)^{1/2}$

$= 2\pi x \left(1 + (1 + 2\pi \sin(\pi x)\cos(\pi x))^2\right)^{1/2} dx$

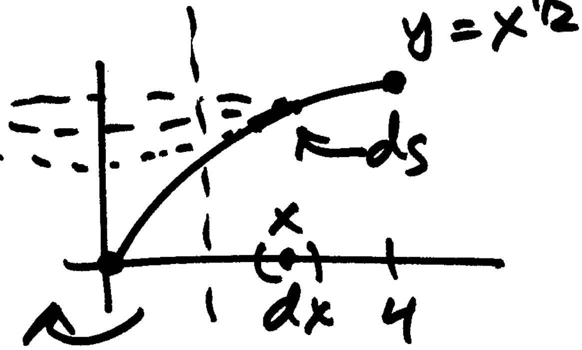
$= \left(1 + \left[\frac{dy}{dx}\right]^2\right)^{1/2} dx$

$$S = \int_0^2 2\pi x \,(\text{etc} \ldots)\, dx$$

$\frac{dy}{dx} = 1 + 2\sin(\pi x) \cdot \cos(\pi x) \cdot \pi$

$= 1 + 2\pi \sin(\pi x)\cos(\pi x)$

e.g. Consider



Want surface area.

x-integral

$$ds = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$

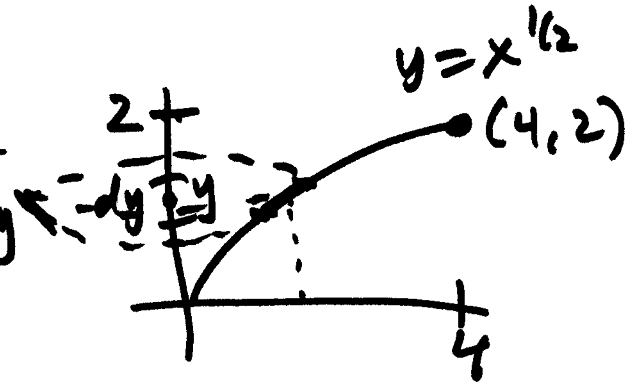
$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$dS = 2\pi x\left(1+\frac{1}{4x}\right)^{1/2} dx$$

$$S = \int_0^4 2\pi x\left(1+\frac{1}{4x}\right)^{1/2} dx$$

y-integral.

$$dS = 2\pi y^2\left((2y)^2+1\right)^{1/2} dy$$



$$S = \int_0^2 2\pi y^2 (4y^2+1)^{1/2} dy$$

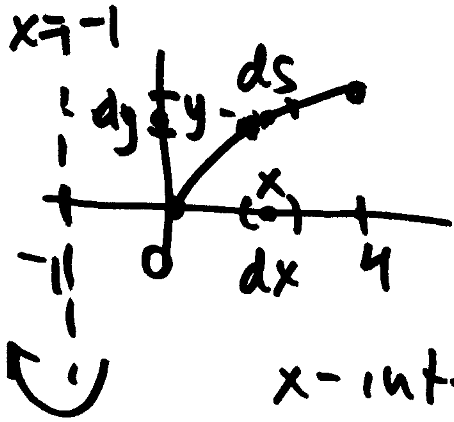
$y = x^{1/2}$ $\quad \underline{x = y^2}$

$$ds = (dx^2+dy^2)^{1/2}$$

$$= \left(\left(\frac{dx}{dy}\right)^2+1\right)^{1/2} dy$$

$$\frac{dx}{dy} = 2y$$

About $x = -1$



x-integral: $dS = 2\pi(x+1)\left(1+\frac{1}{4x}\right)^{1/2} dx$

$$S = \int_0^4 dS = \int_0^4$$

y-integral: $dS = 2\pi(y^2+1)(4y^2+1)^{1/2} dy$

8.4 32) 17) 23)

8.1 21) 65) 11) 25) 37) 49) 27) 55)

7.4 77)

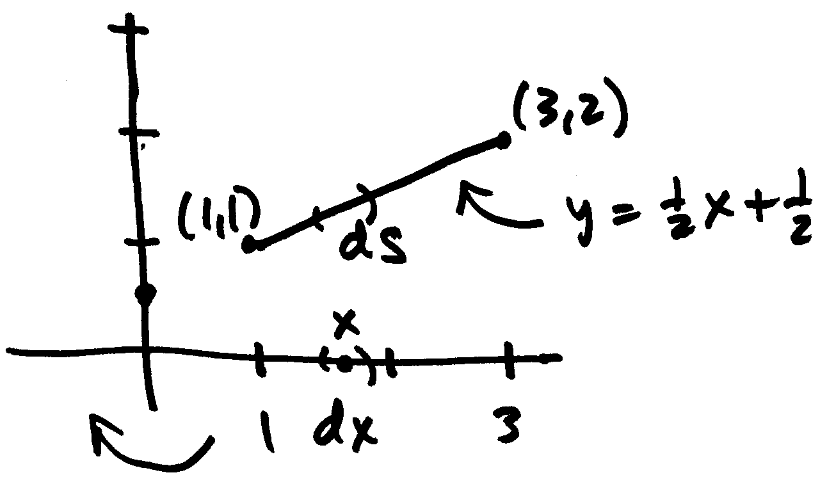
7.2 5)

8.2 29) 39)

8.3 35) 19)

6.5 12) 15)

6.5 12) $y = \frac{1}{2}x + \frac{1}{2}$ $\quad 1 \le x \le 3$



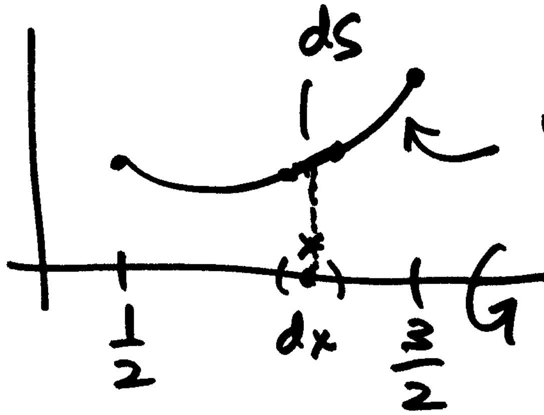
$$ds = (dx^2 + dy^2)^{1/2}$$
$$= \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$
$$\frac{dy}{dx} = \frac{1}{2}$$

$$dS = 2\pi x \left(1 + \left(\tfrac{1}{2}\right)^2\right)^{1/2} dx$$
$$= 2\pi x \sqrt{\tfrac{5}{4}}\, dx$$
$$= \sqrt{5}\,\pi x\, dx$$

$$S = \int_1^3 \sqrt{5}\,\pi x\, dx = \sqrt{5}\,\pi \int_1^3 x\, dx = \sqrt{5}\,\pi \left.\tfrac{1}{2}x^2\right|_1^3$$
$$= \sqrt{5}\,\pi\left(\frac{9}{2} - \frac{1}{2}\right) = 4\sqrt{5}\,\pi$$

Geometry: $\pi(1+3)(2^2+1^2)^{1/2} = 4\pi\sqrt{5}$.

15) $y = (2x - x^2)^{1/2} \qquad \frac{1}{2} \le x \le \frac{3}{2} \qquad$ x-axis



$y = (2x - x^2)^{1/2}$

$$dS = 2\pi (2x - x^2)^{1/2}\, ds$$

$$= 2\pi (2x - x^2)^{1/2} \left(\frac{1}{2x - x^2}\right)^{1/2} dx$$

$$= 2\pi\, dx$$

$$S = \int_{\frac{1}{2}}^{\frac{3}{2}} 2\pi\, dx$$

$$ds = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$

$$\frac{dy}{dx} = \frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x)$$

$$= (2x - x^2)^{-1/2}(1 - x)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (2x - x^2)^{-1}(1 - x)^2$$

$$= 1 + \frac{(1 - x)^2}{2x - x^2}$$

$$= \frac{\cancel{2x} - \cancel{x^2} + 1 - \cancel{2x} + \cancel{x^2}}{2x - x^2}$$

$$= \frac{1}{2x - x^2}$$

7.2 5) $\boxed{\frac{dL}{dx} = -kL}$ $L(x)$ = light at depth x feet.

$\int \frac{dL}{L} = \int -k\,dx$

$\ln(L) = -kx + C$

$e^{\ln(L)} = e^{-kx+C}$

$L(x) = \underbrace{e^C}_{\text{const.}} e^{-kx} = \cancel{\ } = L_0 e^{-kx}$

$L_0 = L(0)$ = light intensity at surface.

Problem: find k (sometimes find L_0)

$\frac{L_0}{2} = L(18) = L_0 e^{-18k}$

$e^{-18k} = \frac{1}{2} \rightarrow -18k = \ln(\frac{1}{2})$

$k = \frac{-\ln(\frac{1}{2})}{18} = \frac{\ln 2}{18} \approx .039$

Solve $L(x) = \frac{1}{10}L_0$ for x

$L_0 e^{-.039x} = \frac{1}{10}L_0$

$e^{-.039x} = \frac{1}{10}$

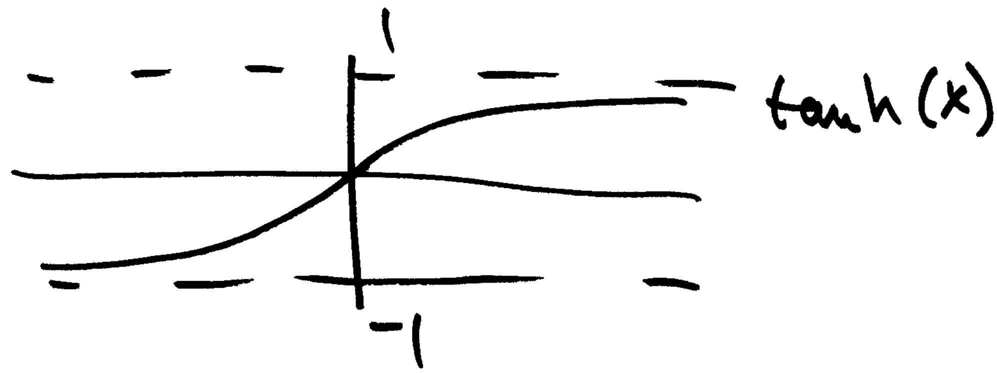
$-.039x = \ln(\frac{1}{10})$

$x = \frac{\ln(\frac{1}{10})}{-.039} \approx 59.8$

$\boxed{\text{About 60 ft}}$

7.4 77) $m \frac{dv}{dt} = mg - kv^2$ v = velocity, t = time

Solution: $v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}}\, t\right)$ etc....



8.1 11) $\int e^{\theta} \csc(e^{\theta}+1)\, d\theta$ $u = e^{\theta}+1$, $du = e^{\theta}\, d\theta$

$= \int \csc(u)\, du$ etc...

21) $\int 3^{x+1}\, dx = \int 3^x \cdot 3^1\, dx = 3 \underbrace{\int 3^x\, dx}$

$= 3 \int e^{x \ln(3)}\, dx$

$\boxed{3^x = e^{\ln(3^x)} = e^{x \ln(3)}}$

$= 3 \cdot \frac{1}{\ln(3)} e^{x \ln(3)} + C$

$= \frac{3}{\ln(3)} 3^x + C$

$\boxed{\text{NO! } \frac{1}{x+2} 3^{x+2} + C \text{ NO! NO!}}$

25) $\int \frac{9}{1+9u^2}\,du$

$$\boxed{\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C}$$

One way: use $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$$= \int \frac{\cancel{9}}{\cancel{9}\left(\frac{1}{9}+u^2\right)}\,du = \int \frac{du}{\frac{1}{9}+u^2} = 3\tan^{-1}(3u) + c$$

$\left(\frac{1}{3}\right)^2$ so $a = \frac{1}{3}$

Another way: Substitute so looks like this

$$\int \frac{9}{1+9u^2}\,du = \int \frac{9}{1+(3u)^2}\,du \qquad \begin{aligned} x &= 3u \\ dx &= 3\,du \end{aligned}$$

$$= 3\int \frac{3\,du}{1+(3u)^2} = 3\int \frac{dx}{1+x^2} = 3\tan^{-1}(x) + c$$

$$= 3\tan^{-1}(3u) + C.$$

27) $\int_0^{\frac{1}{6}} \frac{dx}{\sqrt{1-9x^2}} = \int_0^{\frac{1}{6}} \frac{dx}{\sqrt{9\left(\frac{1}{9}-x^2\right)}} = \frac{1}{3}\int_0^{\frac{1}{6}} \frac{dx}{\sqrt{\frac{1}{9}-x^2}}$

$a = \frac{1}{3}$

$$= \frac{1}{3}\sin^{-1}(3x)\Big|_0^{\frac{1}{6}} = \frac{1}{3}\left(\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}(0)\right) = \frac{\pi}{18}$$

37) $$\int_1^2 \frac{8\,dx}{x^2-2x+2} = \int_1^2 \frac{8\,dx}{(x-1)^2+1}$$

$$(x^2-2x+1)+2-1$$
$$(x-1)^2+1$$

$$= 8\cdot \tan^{-1}(x-1)\Big|_1^2$$

etc...

49) $$\int_{\sqrt{2}}^3 \frac{2x^3}{x^2-1}\,dx$$

$$\begin{array}{r l}
 & 2x + \frac{2x}{x^2-1} \\
x^2-1 \,\big) & 2x^3+0x^2+0x+0 \\
 & -(2x^3 \quad\quad -2x) \\ \hline
 & 2x+0
\end{array}$$

$$= \int_{\sqrt{2}}^3 \left(2x + \frac{2x}{x^2-1}\right) dx$$

$$\int_{\sqrt{2}}^3 \frac{2x}{x^2-1}\,dx$$

$u = x^2-1$

$du = 2x\,dx$

$x=\sqrt{2}$ $u=1$

$x=3$ $u=8$

$$= \int_1^8 \frac{du}{u} \quad \text{etc...}$$

$$= \int_{\sqrt{2}}^3 2x\,dx + \int_{\sqrt{2}}^3 \frac{2x}{x^2-1}\,dx$$

$$= \int_{\sqrt{2}}^3 2x\,dx + \int_1^8 \frac{du}{u} = \cdots$$

55) $$\int_0^{\pi/4} \frac{1+\sin(x)}{\cos^2(x)}\,dx = \int_0^{\pi/4}\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)dx$$

$$= \underbrace{\int_0^{\pi/4} \sec^2(x)\,dx}_{\tan(x)\Big|_0^{\frac{\pi}{4}}} + \underbrace{{}^{-}\int_0^{\pi/4} \frac{-\sin(x)}{\cos^2(x)}\,dx}$$

$$\tan(x)\Big|_0^{\frac{\pi}{4}}$$

$+$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$x = 0 \quad u = 1$

$x = \frac{\pi}{4} \quad u = \frac{1}{\sqrt{2}}$

$$\underbrace{-\int_1^{\frac{1}{\sqrt{2}}} u^{-2}\,du} = \int_{\frac{1}{\sqrt{2}}}^{1} u^{-2}\,du$$

8.2 29) $\int \sin(\ln x)\,dx$ $\quad u = \ln x \quad x = e^u$

$du = \frac{1}{x}\,dx$

$= \int \sin(u)\,e^u\,du$ $\quad dx = x\,du$

$= e^u\,du$

$= \int e^u \sin(u)\,du$

etc....

39) $\int x^n \cos(x)\,dx$ $\quad u = x^n \quad dv = \cos(x)dx$

$du = nx^{n-1}dx \quad v = \sin(x)$

$= x^n \sin(x) - \int n x^{n-1} \sin(x)\,dx$

8.3 19) $\int \frac{dx}{(x^2-1)^2} = \int \frac{dx}{((x+1)(x-1))^2} = \int \frac{dx}{(x+1)^2(x-1)^2}$

$$\frac{1}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

etc...

35) $\cancel{\int \frac{e^t\,dt}{e^{2t}+3e^t+2}}$ $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$

$u = e^t$
$du = e^t\,dt$

$$= \int \frac{du}{u^2+3u+2} \quad \text{etc...}$$

$$\frac{1}{u^2+3u+2} = \frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}$$

clear fractions:

$1 = A(u+2) + B(u+1)$ $\quad u=-1$

$1 = A + 0$ $\quad \boxed{A=1}$

$u=-2$

$1 = 0 + (-B)$ $\quad \boxed{B=-1}$

$$= \int \frac{1}{u+1}\,du - \int \frac{1}{u+2}\,du$$

$$= \ln(u+1) - \ln(u+2) + C$$

$$= \ln(e^t+1) - \ln(e^t+2) + C$$

8.4 17) $\int_0^{\pi} \sqrt{1-\sin^2 t}\, dt$

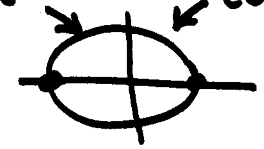
$\cos^2 t + \sin^2 t = 1$

$\cos^2 t = 1 - \sin^2 t$

$= \int_0^{\pi} \sqrt{\cos^2 t}\, dt$

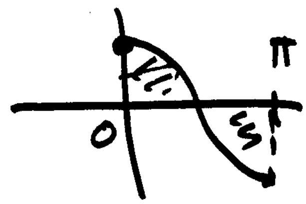
$\sqrt{\cos^2 t} = |\cos t|$

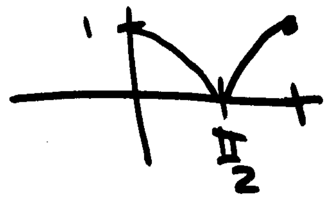
$= \int_0^{\pi} |\cos(t)|\, dt$



$= \int_0^{\pi/2} \cos(t)\, dt - \int_{\frac{\pi}{2}}^{\pi} \cos(t)\, dt$

etc ...





23) $\int_{-\frac{\pi}{3}}^{0} 2\sec^3(x)\, dx$

$\tan^2 x + 1 = \sec^2 x$

$= \int_{-\frac{\pi}{3}}^{0} 2(\tan^2 x + 1)\sec(x)\, dx$

$= 2\int_{-\frac{\pi}{3}}^{0} \tan^2 x \sec(x)\, dx + 2\int_{-\frac{\pi}{3}}^{0} \sec(x)\, dx$

?

By parts:

$$\int_{-\frac{\pi}{3}}^{0} 2\sec^3(x)$$

$u = \sec(x)$ $dv = \sec^2(x)dx$

$du = \tan(x)\sec(x)\,dx$

$v = \tan(x)$

$$= \sec(x)\tan(x)\Big|_{\frac{\pi}{3}}^{\overset{?}{0}} - \int_{-\frac{\pi}{3}}^{0} \tan^2(x)\sec(x)\,dx$$

$$\int \tan^2(x)\sec(x)\,dx = \int (\sec^2(x) - 1)\sec(x)\,dx$$

$$= \int \sec^3(x) - \sec(x)\ dx$$

See example 6 p 568

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32) $\displaystyle\int_{\frac{\pi}{4}}^{\pi/2} 8\cot^4(t)\,dt$

$$\int \tan^4(t)\,dt$$

$$= \int \tan^2(t)(\sec^2(t) - 1)\,dt$$

$$= \int \tan^2 t\sec^2 t\,dt - \int \tan^2 t\,dt$$

$u = \tan t$

$du = \sec^2 t\,dt$

8.5 Trig substitution

Used on integrals involving

$\sqrt{a^2-x^2}$ $\sqrt{x^2-a^2}$ $\sqrt{a^2+x^2}$

Substitutions use identities $\cos^2 x+\sin^2 x=1$

$\tan^2 x+1=\sec^2 x$

$\sqrt{a^2-x^2} \leftarrow x=a\sin\theta$

$\sqrt{x^2-a^2} \leftarrow x=a\sec\theta$

$\sqrt{a^2+x^2} \leftarrow x=a\tan\theta$

e.g. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

$x=2\tan\theta$ $\tan\theta=\frac{x}{2}$ $\theta=\tan^{-1}(\frac{x}{2})$

$dx=2\sec^2\theta\, d\theta$

$\sqrt{4+x^2}=\sqrt{4+(2\tan\theta)^2}$

$=\sqrt{4+4\tan^2\theta}=\sqrt{4(1+\tan^2\theta)}$

$=2\sqrt{1+\tan^2\theta}=2\sqrt{\sec^2\theta}$

$=2\sec\theta.$

$=\displaystyle\int \frac{\cancel{2}\sec^2\theta\, d\theta}{\cancel{2}\sec\theta}$

$=\displaystyle\int \sec\theta\, d\theta$

$=\ln|\sec\theta+\tan\theta|+C$

$=\ln\left|\underbrace{\sec(\tan^{-1}(\frac{x}{2}))}_{*}+\underbrace{\tan(\tan^{-1}(\frac{x}{2}))}_{x/2}\right|+C$

$$\circledast \quad \sec^2\theta = \tan^2\theta + 1$$

$$\sec^2\left(\tan^{-1}\left(\tfrac{x}{2}\right)\right) = \tan^2\left(\tan^{-1}\left(\tfrac{x}{2}\right)\right) + 1$$

$$= \left(\tfrac{x}{2}\right)^2 + 1 \qquad \left[\tan\left(\tan^{-1}\left(\tfrac{x}{2}\right)\right)\right]^2$$

$$\sec\left(\tan^{-1}\left(\tfrac{x}{2}\right)\right) = \sqrt{\tfrac{x^2}{4} + 1}$$

$$= \ln\left|\sqrt{\tfrac{x^2}{4}+1} + \tfrac{x}{2}\right| + C$$

e.g

$$\int \sqrt{1-9t^2}\,dt = \int \sqrt{9\left(\tfrac{1}{9} - t^2\right)}\,dt$$

$$= 3\int \sqrt{\tfrac{1}{9} - t^2}\,dt \qquad t = \tfrac{1}{3}\sin\theta$$

$$dt = \tfrac{1}{3}\cos\theta\,d\theta$$

<u>OR</u> $\int \sqrt{1-9t^2}\,dt = \int \sqrt{1-(3t)^2}\,dt$

$$3t = \sin\theta$$

$$t = \tfrac{1}{3}\sin\theta$$

$$\left(\tfrac{1}{9} - t^2\right)^{1/2} = \left(\tfrac{1}{9} - \tfrac{1}{9}\sin^2\theta\right)^{1/2} = \tfrac{1}{3}\left(1 - \sin^2\theta\right)^{1/2}$$

$$= \tfrac{1}{3}\left(\cos^2\theta\right)^{1/2} = \tfrac{1}{3}\cos\theta$$

$$= 3\int \tfrac{1}{3}\cos\theta \cdot \tfrac{1}{3}\cos\theta\,d\theta$$

$$= \frac{1}{3}\int \cos^2\theta \, d\theta \qquad \cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos(2\theta)$$

$$= \frac{1}{2}(1+\cos(2\theta))$$

$$= \frac{1}{6}\int (1+\cos(2\theta))\,d\theta$$

$$= \frac{1}{6}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \qquad t = \frac{1}{3}\sin\theta$$

$$= \frac{1}{6}\theta + \frac{1}{12}\sin(2\theta) + C \qquad \theta = \sin^{-1}(3t)$$

$$= \frac{1}{6}\sin^{-1}(3t) + \frac{1}{12}\underbrace{\sin(2\cdot\sin^{-1}(3t))}_{} + C$$

use $\sin 2\theta = 2\sin\theta\cos\theta$

$$\frac{1}{12}\cdot 2\sin(\sin^{-1}(3t))\cos(\sin^{-1}(3t))$$

$$= \frac{1}{6}3t\cdot\sqrt{1-\sin^2(\sin^{-1}(3t))}$$

$$= \frac{1}{2}t\sqrt{1-(3t)^2}$$

$$= \frac{1}{6}\sin^{-1}(3t) + \frac{1}{2}t\sqrt{1-9t^2} + C$$

e.g. $\int \frac{dx}{(4-x^2)^{3/2}}$

$x = 2\sin\theta \quad \theta = \sin^{-1}\left(\frac{x}{2}\right)$

$dx = 2\cos\theta \, d\theta$

$(4-x^2)^{3/2} = (4-4\sin^2\theta)^{3/2}$

$= (4(1-\sin^2\theta))^{3/2}$

$= (4\cos^2\theta)^{3/2} = 4^{3/2}\cos^3(\theta)$

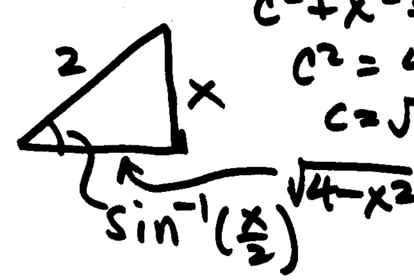
$= \int \frac{2\cos\theta}{4^{3/2}\cos^3\theta} \, d\theta$

$= \frac{2}{8}\int \frac{1}{\cos^2\theta} \, d\theta$

$= \frac{1}{4}\int \sec^2\theta \, d\theta = \frac{1}{4}\tan\theta + C$

$= \frac{1}{4}\tan\left(\sin^{-1}\left(\frac{x}{2}\right)\right) + C$

$= \frac{1}{4} \cdot \frac{x}{\sqrt{4-x^2}} + C$



$c^2 + x^2 = 2^2$

$c^2 = 4 - x^2$

$c = \sqrt{4-x^2}$

OR

$\tan\left(\sin^{-1}\left(\frac{x}{2}\right)\right) = \frac{\sin\left(\sin^{-1}\left(\frac{x}{2}\right)\right)}{\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)}$

$= \frac{\frac{x}{2}}{\sqrt{1-\sin^2\left(\sin^{-1}\left(\frac{x}{2}\right)\right)}}$

$= \frac{\frac{x}{2}}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$

$= \frac{\frac{x}{2}}{\sqrt{1-\frac{x^2}{4}}}$

$= \frac{\frac{x}{2}}{\sqrt{\frac{1}{4}(4-x^2)}}$

$= \frac{\frac{x}{2}}{\frac{1}{2}\sqrt{4-x^2}}$

$= \frac{x}{\sqrt{4-x^2}}$

e.g. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}$

$x = \sec\theta \qquad \theta = \sec^{-1}(x)$

$dx = \sec\theta\tan\theta\,d\theta$

$(x^2-1)^{5/2} = (\sec^2\theta - 1)^{5/2}$

$= (\tan^2\theta)^{5/2} = \tan^5\theta$

$$= \int \frac{\sec^2\theta\cdot\sec\theta\tan\theta}{\tan^5\theta}\,d\theta$$

$$= \int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

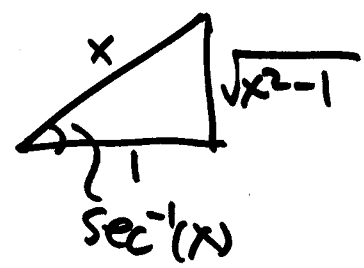
$$= \int \frac{1}{\cos^3\theta}\cdot\frac{\cos^4\theta}{\sin^4\theta}\,d\theta = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \int \frac{du}{u^4} = \int u^{-4}\,du = -\frac{1}{3}u^{-3} + c$$

$$= -\frac{1}{3}\,\frac{1}{\sin^3\theta} + c$$



$$= -\frac{1}{3}\,\frac{1}{\sin^3(\sec^{-1}(x))} + c$$

$\sin(\sec^{-1}(x)) = \dfrac{\sqrt{x^2-1}}{x}$

$$= \frac{-1}{3}\,\frac{1}{\left[\dfrac{(x^2-1)^{1/2}}{x}\right]^3} + c = -\frac{1}{3}\,\frac{x^3}{(x^2-1)^{3/2}} + c$$

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