MAPLE #1 due tomorrow. AT 9:45

$f(x) = x^{\#} + \sin^2(\pi x)$ sin(Pi*x)^2

~~sin^2(Pi*x)~~

sin(3.14159*x)^2
can use this if necessary

Exam 2 - Friday 6.5, 7.2, 7.4, 8.1–8.4.
Can bring in 3x5 card as well

8.3 Partial fractions

This technique is usually applied to rational functions, i.e., $\frac{f(x)}{g(x)}$, f and g are polynomials.

e.g. $$\int \frac{5x-3}{x^2-2x-3}\,dx$$

$$= \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln(x+1) + 3\ln(x-3) + C$$

$x^2-2x-3 = (x+1)(x-3)$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$$

verify:

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)+3(x+1)}{(x+1)(x-3)}$$

$$= \frac{5x-3}{(x+1)(x-3)} \checkmark$$

How did we do it?

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}.$$ Find A and B.

<u>Method 1</u>: Undetermined coefficients.

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

$$= \frac{(A+B)x+(-3A+B)}{(x+1)(x-3)}$$

$$3(A+B=5)$$
$$-3A+B=-3$$
$$4B=12$$
$$\boxed{B=3}$$

$$A+3=5$$
$$\boxed{A=2}$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$$

$$-(A+B=5)$$
$$-3A+B=-3$$
$$-4A=-8$$
$$\boxed{A=2}$$
$$2+B=5$$
$$\boxed{B=3}$$

$\underline{\text{Method 2}}$: $\left[\dfrac{5x-3}{(x+1)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x-3}\right](x+1)(x-3)$

$$5x-3 = A(x-3) + B(x+1) \qquad : x=-1$$

$$5(-1)-3 = A(-1-3) + B\cancelto{0}{(-1+1)}$$

$$-8 = -4A \qquad \boxed{A=2}$$

$$5(3)-3 = A\cancelto{0}{(3-3)} + B(3+1) \qquad : x=3$$

$$12 = 4B \qquad \boxed{B=3}$$

~~[illegible]~~

$\underline{\text{eg}}$ #2 $\qquad \dfrac{5x-7}{x^2-3x+2} = \dfrac{5x-7}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$

$$\boxed{x^2-3x+2 = (x-1)(x-2)}$$

$$5x-7 = A(x-2) + B(x-1)$$

$\underline{x=1}$

$$5-7 = A(1-2) + 0$$

$$-2 = -A \qquad \boxed{A=2}$$

$\underline{x=2}$

$$10-7 = A\cancelto{0}{(2-2)} + B(2-1)$$

$$3 = B \qquad \boxed{B=3} \qquad \frac{5x-7}{x^2-3x+2} = \frac{2}{x-1} + \frac{3}{x-2} \,//$$

e.g. $$\frac{z+1}{z^3 - z^2 - 6z} = \frac{z+1}{z(z+2)(z-3)}$$

$$z^3 - z^2 - 6z = z(z^2 - z - 6) = z(z+2)(z-3)$$

$$= \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z-3}$$

Clear fractions

$$\left[\frac{z+1}{z(z+2)(z-3)} = \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z-3}\right] z(z+2)(z-3)$$

$$z + 1 = A(z+2)(z-3) + Bz(z-3) + C(z)(z+2)$$

$$1 = -6A + 0 + 0 \qquad \boxed{A = -\frac{1}{6}} \qquad \underline{z = 0}$$

$$\underline{z = 3}$$

$$4 = 15C + 0 + 0 \qquad \boxed{C = \frac{4}{15}}$$

$$\underline{z = -2}$$

$$-1 = 0 + 10B + 0 \qquad \boxed{B = -\frac{1}{10}}$$

Repeated linear factors

e.g. $\int \frac{6x+7}{(x+2)^2}\,dx = \int \frac{6}{x+2}\,dx - \int \frac{5}{(x+2)^2}\,dx$

$$\left[\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}\right](x+2)^2$$

Clear fractions.

$6x+7 = A(x+2) + B$

$x=-2$

$-12+7 = 0 + B$ $\quad \boxed{B=-5}$

$x=0$

$7 = 2A + B$

$7 = 2A - 5$ $\quad \boxed{A=6}$

$12 = 2A$

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}$$

$$= 6\ln(x+2) - \underbrace{\int \frac{5}{(x+2)^2}\,dx}_{*}$$

(*) $\int \frac{5}{(x+2)^2}dx$ $\quad u = x+2$, $du = dx$

$= \int \frac{5}{u^2}du = \int 5u^{-2}du = -5u^{-1}$

$= -5(x+2)^{-1}$

$= 6\ln(x+2) + \frac{5}{(x+2)} + c$

e.g. $\int \frac{2x^2+x+1}{x(x+1)^2}dx$

$\left[\frac{2x^2+x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right] x(x+1)^2$

Clear fractions

$2x^2+x+1 = A(x+1)^2 + Bx(x+1) + Cx$

$x = -1$

$2-1+1 = 0+0-C$ $\quad \boxed{C=-2}$

$2 = -C$

$x = 0$

$1 = A + 0 + 0$ $\quad \boxed{A=1}$

$2+1+1 = 4A + 2B + C$ $\quad\quad$ $\underline{x=1}$

$= 4 + 2B - 2$ $\quad$ $\boxed{B=1}$

$\cancel{4} = \cancel{4} + 2B - 2$

$2B = 2$

etc ...

irreducible quadratic factors

e.g. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t(t^2+1)}\,dt$

$t(t^2+1)$ is already in lowest form

t^2+1 cannot be factored. It is irreducible.

$$\left[\frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1}\right] t(t^2+1)$$

clear fractions.

$3t^2+t+4 = A(t^2+1) + (Bt+C)t$

$4 = A$ $\quad\quad$ $\boxed{A=4}$ $\quad\quad$ $t=0$

Trick: Take the derivative!

$$6t+1 = A\cdot 2t + (Bt+C) + Bt$$

$t=0$

$$1 = 0 + C + 0 \qquad \boxed{C=1}$$

Do it again!

$$6 = 2A + 2B \qquad \boxed{B=-1}$$

$$6 = 8 + 2B$$

$$-2 = 2B$$

$$= \int_1^{\sqrt{3}} \frac{4}{t}\,dt + \int_1^{\sqrt{3}} \frac{-t+1}{t^2+1}\,dt$$

$$= \underbrace{\int_1^{\sqrt{3}} \frac{4}{t}\,dt}_{\ln t} - \underbrace{\int_1^{\sqrt{3}} \frac{t}{t^2+1}\,dt}_{\substack{\text{subst} \\ u=t^2+1}} + \underbrace{\int_1^{\sqrt{3}} \frac{1}{t^2+1}\,dt}_{\tan^{-1}(t)}$$

etc...

e.g. #30 p563

$$\int \frac{x^4}{x^2-1}\,dx$$

$$\begin{array}{r}
x^2+1 \quad + \quad \frac{1}{x^2-1} \\
x^2-1 \,\big|\overline{\, x^4+0x^3+0x^2+0x+0} \\
-(x^4+0x^3-x^2) \\
\hline
x^2+0x+0 \\
-(x^2+0x-1) \\
\hline
1
\end{array}$$

$$= \int \left(x^2+1 + \frac{1}{x^2-1}\right) dx$$

$$= \int (x^2+1)\,dx + \int \frac{1}{x^2-1}\,dx$$

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

clear fractions

$$1 = A(x-1) + B(x+1) \qquad x=-1$$

$$1 = -2A + 0 \qquad \boxed{A = -\tfrac{1}{2}}$$

$$1 = 2B \qquad \boxed{B = \tfrac{1}{2}} \qquad x=1$$

$$= \int (x^2+1)\,dx - \int \frac{1}{2(x+1)}\,dx + \int \frac{1}{2(x-1)}\,dx$$

$$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + C$$

e.g. $\displaystyle\int \frac{t^4+9}{t^4+9t^2}\,dt$

$$\frac{t^4+9}{t^4+9t^2} = \frac{(t^4+9t^2)+(9-9t^2)}{t^4+9t^2} = 1 + \frac{9-9t^2}{t^4+9t^2}$$

$$= 1 + 9\,\frac{1-t^2}{t^2(t^2+9)}$$

$$\left[\frac{1-t^2}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}\right] t^2(t^2+9)$$

Clear fractions

$$1-t^2 = At(t^2+9) + B(t^2+9) + (Ct+D)\cdot t^2$$

$\underline{t=0}$

$$1 = 0 + 9B + 0 \qquad \boxed{B = \frac{1}{9}}$$

Derivative:

$$-2t = At\cdot 2t + A(t^2+9) + 2tB + 2t(Ct+D) + Ct^2$$

$\underline{t=0}$

$$0 = 0 + 9A + 0 + 0 + 0 \qquad \boxed{A=0}$$

Derivative:

$$-2 = 4At + 2At + 2B + 2t\cdot C + 2(Ct+D) + 2tC$$

$\underline{t=0}$

$$-2 = 0 + 0 + 2B + 0 + 2D + 0$$

$-2 = 2B + 2D$ $\boxed{D = -\frac{10}{9}}$

$-2 = \frac{2}{9} + 2D$

$-18 = 2 + 18D$

$18D = -20$

$D = -\frac{20}{18} = -\frac{10}{9}$

Derivative:

$0 = 4A + 2A + 0 + 2C + 2C + 2C$

$0 = 6A + 6C = 6(A+C)$ $\boxed{C=0}$

$0 = A + C$

$\therefore \frac{1-t^2}{t^2(t^2+9)} = \frac{1}{9t^2} - \frac{10}{9(t^2+9)}$

$$\frac{t^4+9}{t^4+9t^2} = 1 + 9\left(\frac{1}{9t^2} - \frac{10}{9(t^2+9)}\right)$$

$$= 1 + \frac{1}{t^2} - \frac{10}{t^2+9}$$

$$\int \frac{t^4+9}{t^4+9t^2}\,dt = \int dt + \int \frac{1}{t^2}\,dt - 10\int \frac{1}{t^2+9}\,dt$$

etc....

8.4 Trigonometric Integrals

1. $\int \sin^m(x)\cos^n(x)\,dx$ m, n integers (positive or negative)

a. If at least one of m or n is odd

e.g. $\int \sin^3(x)\cos^2(x)\,dx$

$= \int \sin^2(x)\cos^2(x)\sin(x)\,dx$

$\cos^2(x)+\sin^2(x)=1$

$\sin^2(x)=1-\cos^2(x)$

$= \int (1-\cos^2(x))\cos^2(x)\sin(x)\,dx$

$u=\cos(x)$

$du=-\sin(x)\,dx$

$= -\int (1-u^2)u^2\,du$

$= \int (u^2-1)u^2\,du = \int (u^4-u^2)\,du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$

e.g. $\int \sin^2(x)\cos^3(x)\,dx$

$= \int \sin^2(x)\cos^2(x)\,\cos(x)\,dx$

$\cos^2(x) = 1-\sin^2(x)$

$= \int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

$u = \sin(x)$

$du = \cos(x)\,dx$

$= \int u^2(1-u^2)\,du$ etc...

e.g. $\int \sin^3(x)\cos^5(x)\,dx$

$= \int \sin^3(x)\cos^4(x)\,\cos(x)\,dx$

$\cos^2(x) = 1-\sin^2(x)$

$= \int \sin^3(x)\left(1-\sin^2(x)\right)^2 \cos(x)\,dx$

$u = \sin(x)$

$du = \cos(x)\,dx$

$= \int u^3(1-u^2)^2\,du$ = etc...

b. m, n both even.

e.g. $\int \sin^4(x)\cos^2(x)\,dx$

$$\int \underbrace{\sin^3(x)\cos^2(x)}\,\sin(x)\,dx$$

$\sin^2(x) = 1 - \cos^2(x)$

$\sin^3(x) = (1-\cos^2(x))^{3/2}$

$$= \int (1-\cos^2(x))^{3/2}\cos^2(x)\sin(x)\,dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$= \int (1-u^2)^{3/2} u^2\,du \quad ????? \quad \text{NO GOOD.}$$

Use trig identity

$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) \qquad \cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$$

$$= \int \frac{1}{8} - \frac{1}{8}\cos(2x) - \frac{1}{8}\cos^2(2x) + \frac{1}{8}\cos^3(2x)\, dx$$

$$\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$$

$$= \int \frac{1}{16} - \frac{1}{8}\cos(2x) - \frac{1}{16}\cos(4x) + \frac{1}{8}\cos^3(2x)\, dx$$

This is <u>doable</u>.

2. $\int \tan^n(x)\, dx$, $\int \sec^n(x)\, dx$

<u>e.g.</u> $\int \tan^4(x)\, dx$ $\qquad \tan^2(x) = \sec^2(x) - 1$

$$= \int \tan^2(x)\, \underbrace{\tan^2(x)}\, dx$$

$$= \int \tan^2(x)\left(\sec^2(x) - 1\right) dx$$

$$= \int \tan^2(x)\sec^2(x)\, dx - \int \tan^2(x)\, dx$$

$$\int \tan^2(x)\sec^2(x)\,dx \qquad u = \tan(x)$$

$$du = \sec^2(x)\,dx$$

$$= \int u^2\,du = \text{etc} \ldots$$

$$\int \tan^2(x)\,dx = \int (\sec^2(x) - 1)\,dx$$

$$= \tan(x) - x + C$$

Tomorrow go to INNOVATION 105