Exam Solutions are on-line

Wednesday we are in INNOVATION 103

MAPLE #1 due THURSDAY

8.1 Basic Integration Formulas

Substitution

#2 p542

$$\int \frac{3\cos(x)}{\sqrt{1+3\sin(x)}}\,dx \qquad u = 1+3\sin(x)$$
$$du = 3\cos(x)\,dx$$

$$= \int \frac{du}{u^{1/2}} = \int u^{-1/2}\,du = 2u^{1/2} + C$$

$$= 2\left(1+3\sin(x)\right)^{1/2} + C$$

#6 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 z}{\tan z}\,dz$

$u = \tan(z)$

$du = \sec^2(z)\,dz$

$z = \frac{\pi}{4} \quad u = \tan\left(\frac{\pi}{4}\right) = 1$

$z = \frac{\pi}{3} \quad u = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$$= \int_1^{\sqrt{3}} \frac{du}{u}$$

$$= \ln(u)\Big|_1^{\sqrt{3}} = \ln(\sqrt{3}) - \cancel{\ln(1)}^{\,0} = \ln(\sqrt{3}) = \frac{1}{2}\ln(3)$$

#8) $\int \frac{dx}{x-\sqrt{x}}$

$u = x^{1/2} \qquad x = u^2$

$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\underbrace{x^{1/2}}}dx$

$= \frac{1}{2u}dx$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$du = \frac{1}{2}x^{-1/2}dx$

$dx = 2u\,du$

$= \int \frac{2u\,du}{u^2-u}$

$= \int \frac{2\cancel{u}\,du}{\cancel{u}(u-1)} = \int \frac{2\,du}{u-1} \qquad w = u-1, \quad dw = du$

$= \int \frac{2\,dw}{w} = 2\ln(w) + C$

$= 2\ln(u-1) + C = 2\ln(x^{1/2}-1) + C$ //

$\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)} \qquad u = x^{1/2}-1$

etc...

#12) $\int \frac{\cot(3+\ln x)}{x}dx \qquad u = 3+\ln x$

du $\qquad du = \frac{1}{x}dx$

$= \int \cot(u)\,du = \int \frac{\cos(u)}{\sin(u)}du \qquad w = \sin(u), \quad dw = \cos(u)\,du$

$= \int \frac{dw}{w} = \ln(w) + C = \ln(\sin(u)) + C = \underline{\ln(\sin(3+\ln x))} + C$

#44) $\frac{1}{2}\int 2x \sec(x^2-5)\,dx$

$u = x^2-5$

$du = 2x\,dx$

$= \frac{1}{2}\int \sec(u)\,du = \frac{1}{2}\ln(\sec(u)+\tan(u)) + c$

$= \frac{1}{2}\ln(\sec(x^2-5)+\tan(x^2-5)) + c$

$\int \sec(x)\,dx = \int \frac{\sec(x)\tan(x)+\sec^2(x)}{\sec(x)+\tan(x)}\,dx$

$\boxed{\sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)} = \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}}$

$u = \sec(x)+\tan(x)$

$du = (\sec(x)\tan(x)+\sec^2(x))\,dx$

$= \int \frac{du}{u} = \ln(u) + c$

$= \ln(\sec(x)+\tan(x)) + c.$

#34) $\int \frac{dy}{\sqrt{e^{2y}-1}}$

$u = e^{2y}-1$

$du = 2e^{2y}\,dy$

$= 2(u+1)\,dy$

$dy = \frac{du}{2(u+1)}$

$= \int \frac{du}{2(u+1)u^{1/2}}$

$$= \int \frac{dw}{w^2+1}$$

$w = u^{1/2}$ $\quad u = w^2$

$dw = \frac{1}{2} u^{-1/2}\, du$

$= \frac{1}{2u^{1/2}}\, du$

$$= \tan^{-1}(w) + c$$

$$= \tan^{-1}(u^{1/2}) + c$$

$$= \tan^{-1}\left(\sqrt{e^{2y}-1}\right) + c \quad //$$

Simplification tricks.

#38 $$\int_2^4 \frac{2\,dx}{x^2-6x+10} = \int_2^4 \frac{2\,dx}{(x-3)^2+1}$$

Complete the square:

$$x^2 - 6x + 10 = x^2 - 6x + 9 + 10 - 9$$

$\boxed{(x+a)^2 = x^2 + 2a + a^2}$ $\quad = (x-3)^2 + 1$

$$= 2\tan^{-1}(x-3)\Big|_2^4 = 2\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$$

$$= 2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = 2\,\frac{\pi}{2} = \pi \quad //$$

#40) $$\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}} = \sin^{-1}(\theta-1)+c$$

$$2\theta-\theta^2 = -(\theta^2-2\theta+1)+1$$
$$= -(\theta-1)^2+1 = 1-(\theta-1)^2$$

#50 $$\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx = \int_{-1}^{3}\left(2x-3+\frac{2}{2x+3}\right)dx$$

$$\begin{array}{r l}
 & 2x-3 \;+\; \frac{2}{2x+3} \\
2x+3 \,\big) & 4x^2+0x-7 \\
 & -(4x^2+6x) \\
\hline
 & -6x-7 \\
 & -(-6x-9) \\
\hline
 & 2
\end{array}$$

$$= \underbrace{\int_{-1}^{3}(2x-3)\,dx}_{x^2-3x\Big|_{-1}^{3}} + \underbrace{\int_{-1}^{3}\frac{2}{2x+3}\,dx}_{+\int_1^9 \frac{du}{u}}$$

$u=2x+3$

$du=2\,dx$

$x=-1 \quad u=1$

$x=3 \quad u=9$

$$=(9-9-(1+3)) + \ln(u)\Big|_1^9 = -4+\ln(9)-\cancel{\ln(1)}^{0}$$
$$= -4+\ln(9) \;//$$

8.2 Integration by Parts

Idea: Substitution $\longleftrightarrow$ Chain rule

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad u=g(x)$$

Integration by parts $\longleftrightarrow$ Product rule

Product rule: $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

OR $$d(uv) = u\,dv + v\,du$$

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$$\int d(uv) = \int u\,dv + \int v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

e.g. $$\int x\cos(x)\,dx \qquad u=x \quad dv=\cos(x)\,dx$$

$$du=dx \quad v=\sin(x)$$

$$= x\sin(x) - \int \sin(x)\,dx$$

$$= x\sin(x) + \cos(x) + C$$

Note: ① split integrand into u and dv

② choose u so that it simplifies when you differentiate (e.g. a polynomial)

③ choose dv so that it is easy to integrate.

$$\int x\cos(x)\,dx \qquad u=\cos(x) \quad dv = x\,dx$$
$$du = -\sin(x)\,dx \quad v = \tfrac{1}{2}x^2$$

$$= \tfrac{1}{2}x^2\cos(x) + \int \tfrac{1}{2}x^2 \sin(x)\,dx$$

e.g. $\int xe^{-x}\,dx$

$\cancel{u=e^{-x} \quad dv = x\,dx}$

$\cancel{du=-e^{-x}\,dx \quad v=\tfrac{1}{2}x^2}$

$u = x \quad dv = e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

$$= -xe^{-x} + \int e^{-x}\,dx$$
$$= -xe^{-x} - e^{-x} + c$$
$$= -e^{-x}(x+1)$$

CK: $\frac{d}{dx}\left(-e^{-x}(x+1)\right)$

$= -e^{-x} + (x+1)(e^{-x})$

$= -\cancel{e^{-x}} + xe^{-x} + \cancel{e^{-x}}$

$= xe^{-x}$ ✓

e.g. $\int_0^{\pi} x^2 \sin(x)\,dx$ $\quad u = x^2 \quad dv = \sin(x)\,dx$

$du = 2x\,dx \quad v = -\cos(x)$

$$= -x^2\cos(x)\Big|_0^{\pi} + \underbrace{\int_0^{\pi} 2x\cos(x)\,dx}_{*}$$

$*$ $2\int_0^{\pi} x\cos(x)\,dx$ $\quad u = x \quad dv = \cos(x)dx$

etc ....

$$= 2\left(\left[x\sin(x) + \cos(x)\right]\Big|_0^{\pi}\right)$$

$$= 2\left(\pi \underbrace{\cancel{\sin}(\pi)}_{0} + \cos(\pi) - 0\cdot\underbrace{\cancel{\sin}(0)}_{0} - \cos(0)\right)$$

$$= 2(-1-1) = -4$$

$$= -\pi^2\cos(\pi) - (-\underbrace{0^2\cancel{\cos(0)}}_{0}) - 4$$

$$= \pi^2 - 4 \;//$$

e.g. $\int \ln(x)\,dx$ $\qquad u=\ln(x) \quad dv=dx$

$du=\frac{1}{x}dx \quad v=x$

$= x\ln(x) - \int x\cdot\frac{1}{x}dx$

$= x\ln(x) - \int dx = x\ln(x) - x + C$ //

e.g. $\int \sin^{-1}(y)\,dy$ $\qquad u=\sin^{-1}(y) \quad dv=dy$

$du=\frac{1}{\sqrt{1-y^2}}dy \quad v=y$

$= y\sin^{-1}(y) - \underbrace{\int \frac{y}{\sqrt{1-y^2}}dy}_{*} = y\sin^{-1}(y) + (1-y^2)^{1/2} + C$

(*) $\frac{-1}{2}\int \frac{-2y}{\sqrt{1-y^2}}dy$ $\qquad u=1-y^2$

$du=-2y\,dy$

$= -\frac{1}{2}\int \frac{du}{u^{1/2}} = -\frac{1}{2}\int u^{-1/2}du$

$= -\frac{1}{\cancel{2}}(\cancel{2}u^{1/2}) = -u^{1/2} = -(1-y^2)^{1/2}$

e.g. $\int e^x \cos(x)\,dx$ $\quad u = e^x \quad dv = \cos(x)\,dx$

$du = e^x\,dx \quad v = \sin(x)$

$$= e^x \sin(x) - \underbrace{\int e^x \sin(x)\,dx}_{*}$$

$*$ $\quad u = e^x \quad dv = \sin(x)\,dx$

$du = e^x\,dx \quad v = -\cos(x)$

$$= -e^x \cos(x) + \int e^x \cos(x)\,dx$$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx.$$

We have:

$$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x)$$

$$\int e^x \cos(x)\,dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C$$

what if: for $\int e^x \sin(x)\,dx$ we took:

$u = \sin(x) \qquad dv = e^x\,dx$

$du = \cos(x)\,dx \qquad v = e^x$

$= e^x \sin(x) - \int e^x \cos(x)\,dx$

So we get:

$\int e^x \cos(x)\,dx = e^x \cancel{\sin(x) - e^x \sin(x)} + \int e^x \cos(x)\,dx$

So does not work.

eg #30) $\int z(\ln z)^2\,dz$

Subst: $u = \ln z \qquad z = e^u$

$du = \frac{1}{z}\,dz$

$du = \frac{1}{e^u}\,dz$

$dz = e^u\,du$

$= \int e^u \cdot u^2 \cdot e^u\,du$

$= \int u^2 e^{2u}\,du$ etc...

Directly by parts: $u = (\ln z)^2 \quad dv = z\,dz$

$du = 2\ln(z)\cdot\frac{1}{z}dz \quad v = \frac{1}{2}z^2$

$$= \frac{1}{2}z^2(\ln z)^2 - \underbrace{\int z\ln(z)\,dz}_{(*)}$$

(*) $u = \ln z \quad dv = z\,dz$

$du = \frac{1}{z}dz \quad v = \frac{1}{2}z^2$

$$= \frac{1}{2}z^2\ln z - \int \frac{1}{2}z\,dz$$

$$= \frac{1}{2}z^2\ln(z) - \frac{1}{4}z^2$$

$$= \frac{1}{2}z^2(\ln z)^2 - \frac{1}{2}z^2\ln(z) + \frac{1}{4}z^2 + C \quad //$$

TOMORROW
INNOVATION 103