Tuesday we are in ST 1 126

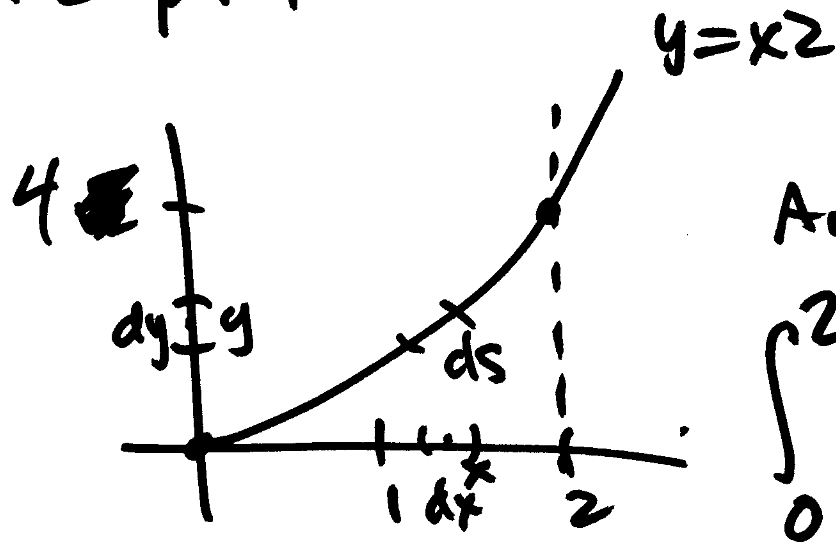
Wednesday we are in INNOVATION 103

MAPLE #1 is AVAILABLE ONLINE

DUE THURSDAY

MAPLE DEMO

#2 p 474



Arclength.

$$\int_0^2 \left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$

$$= \int_0^2 \left(1+(2x)^2\right)^{1/2} dx$$

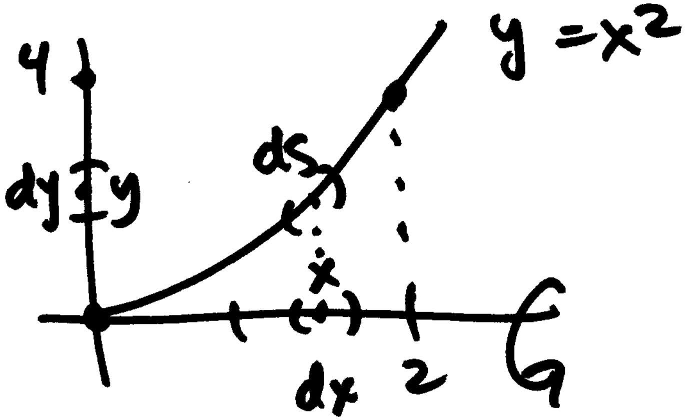
$$= \int_0^2 \left(1+4x^2\right)^{1/2} dx$$

$x = y^{1/2}$

$$\int_0^4 \left(\left(\frac{dx}{dy}\right)^2+1\right)^{1/2} dy$$

$$= \int_0^4 \left(\left(\frac{1}{2}y^{-1/2}\right)^2+1\right)^{1/2} dy$$

$$= \int_0^4 \left(\frac{1}{4y}+1\right)^{1/2} dy.$$



Surface area. (integrate w.r.t. x)

$$\int_0^2 2\pi x^2 \underbrace{\left(1+4(x^2)\right)^{1/2}}_{ds} dx$$

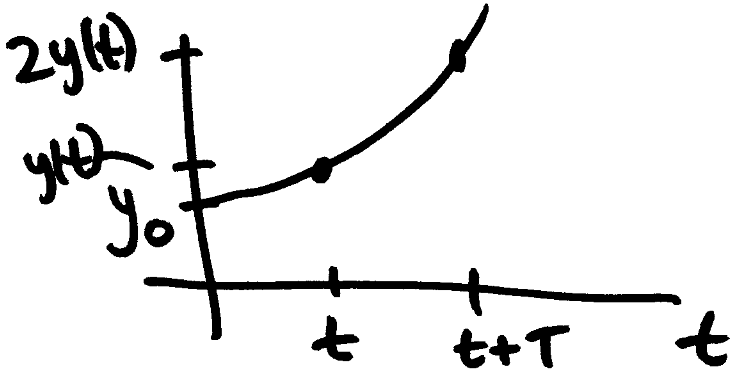
dS

Another way: integrate w.r.t. y

$$\int_0^4 2\pi y \left(\frac{1}{4y}+1\right)^{1/2} dy$$

7.2 Exponential growth / decay

Idea: Solution to differential equation



$$y(t) = y_0 e^{kt}$$

$k > 0$ growth rate

y_0: initial value

1. rapid growth
2. Doubling time is. T is the doubling time if

$$y(t+T) = 2y(t)$$

3. The rate of change $y'(t)$ is proportional to $y(t)$ for each t.

Why is this true? If y' is proportional to $y(t)$,

$$y'(t) = k\,y(t)$$

$$\frac{y'(t)}{y(t)} = k$$

$$\int \frac{y'(t)}{y(t)}\,dt = \int k\,dt$$

$$\int \frac{y'(t)}{y(t)}\,dt \qquad u = y(t) \qquad du = y'(t)\,dt$$

$$= \int \frac{du}{u} = \ln(u) + c$$

$$= \ln(y(t)) + c$$

$$\ln(y(t)) = kt + c$$

$$e^{\ln(y(t))} = e^{kt+c}$$

$$y(t) = e^{c} e^{kt}$$

call this y_0

$$\therefore \boxed{y(t) = y_0 e^{kt}}$$

What is y_0?

$$y(0) = y_0 e^{k \cdot 0} = y_0$$

so y_0 is value of $y(t)$ at $t=0$.

If $k<0$ then we have exponential <u>decay</u>.

<u>e.g</u> #8 p515. Need values for k and $\underline{\underline{y_0}}$.

$$y(t) = y_0 e^{kt}$$

1. Find value for k.

$$y(3) = 10000 \qquad 10000 = y_0 e^{3k}$$
$$y(5) = 40000 \qquad 40000 = y_0 e^{5k}$$

$$4 \cancel{y_0} e^{3k} = \cancel{y_0} e^{5k}$$
$$4 = e^{5k} \cdot e^{-3k} = e^{2k}$$
$$\ln(4) = \ln(e^{2k}) = 2k$$
$$\therefore k = \frac{\ln(4)}{2} = \frac{\ln(2^2)}{2} = \frac{2\ln(2)}{2} = \ln 2 \approxeq .693$$

2. Find y_0.

$$10000 = y_0 e^{3k} = y_0 e^{3\ln(2)} = y_0 e^{\ln(8)} = 8y_0$$

$$\therefore y_0 = \frac{10000}{8} = 1250 //$$

Another look:

$y(3) = 10000$

$y(5) = 40000$

Doubles twice in 2 hrs

So <u>doubles once in 1 hr.</u>

$\therefore$ $y(3) = 10000$

$y(2) = 5000$

$y(1) = 2500$

$y(0) = \underline{1250}$

eg #9) $y(t) = y_0 e^{kt}$

$y(t) = $ # cases of disease

Spse # cases is reduced by 25% per year.

New value of k:

$$y(t) = y_0 e^{kt}$$

$$y(1) = y_0 e^{k} = .75 y_0$$

$$e^{k} = .75$$

$$k = \ln(.75) \approx -.287$$

$$y(t) = 10000 e^{-.287t}$$

$$= 10000 e^{\ln(.75)t}$$

EXAMPLE 1

cases decreases by 20% per year.

This can give us k.

$$y(t) = y_0 e^{kt}$$

$$y(1) = y_0 e^{k} \qquad y(1) = .8 y_0$$

$$\therefore e^{k} = .8$$

$$k = \ln(.8) \approx -.223$$

$$\boxed{y(t) = y_0 e^{-.223t}}$$

cases reduced from 10000 to 1000 in $t \approx 10.32$ yrs

Find t that givies

$$y(t) = 1000$$

$$\therefore 1000 = 10000\, e^{\ln(.75)t}$$

$$\therefore e^{\ln(.75)t} = .1$$

$$\ln(.75)t = \ln(.1)$$

$$\therefore t = \frac{\ln(.1)}{\ln(.75)} \approx 8.0 \text{ yrs.}$$

7.4 Hyperbolic functions.

1. Definitions

Hyperbolic sine, $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

Hyperbolic cosine, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic tangent, $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

etc...

Eulers formula.

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

where $i = \sqrt{-1}$

$$e^{-ix} = \cos(-x) + i\sin(-x)$$
$$= \cos(x) - i\sin(x)$$

$$\boxed{\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}}$$

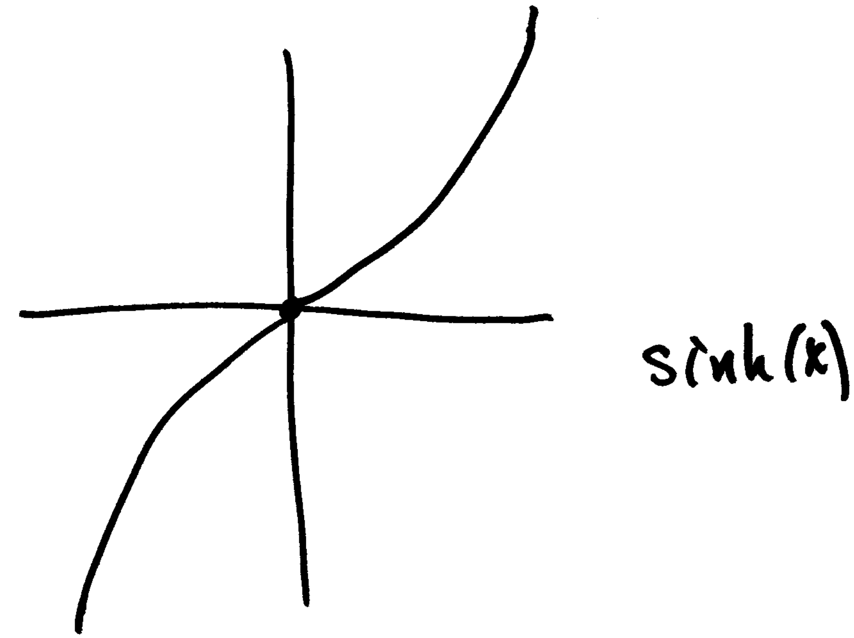
2. Graphs.

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\sinh(-x) = \frac{e^{-x} - e^x}{2}$$

$$= -\sinh(x)$$

$$\sinh(0) = 0$$

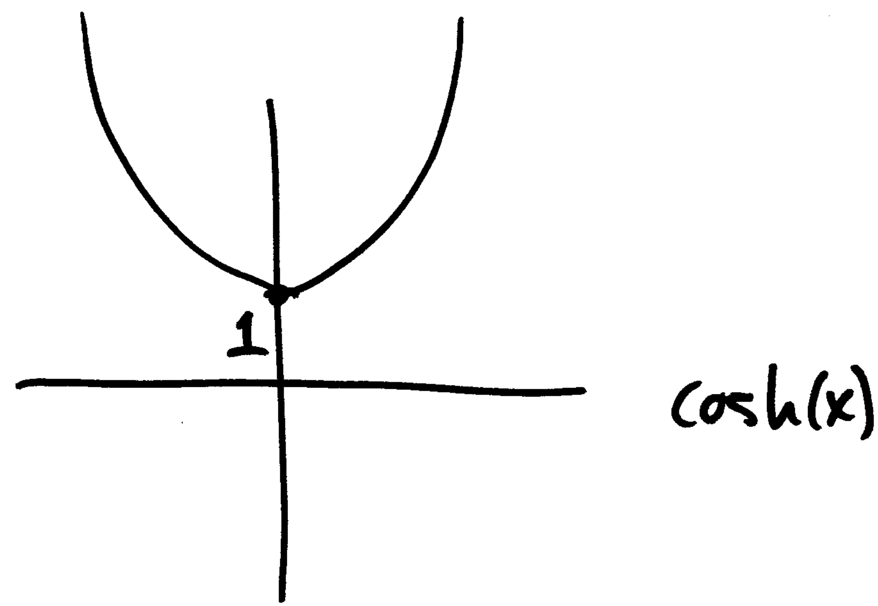


$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

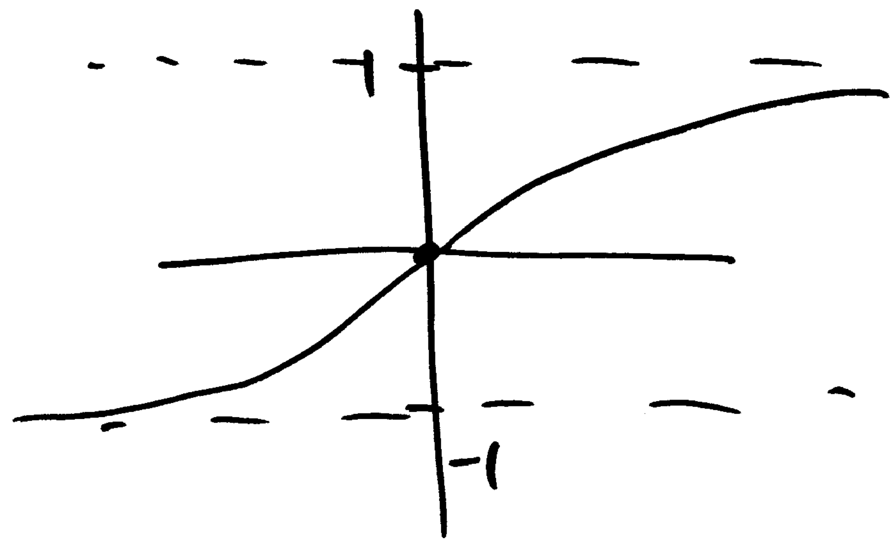
$$\cosh(-x) = \frac{e^{-x} + e^x}{2}$$

$$= \cosh(x)$$

$$\cosh(0) = 1$$



$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$$\tanh(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x}$$

$$= -\tanh(x)$$

$$\tanh(0) = 0$$

$$\lim_{x \to +\infty} \tanh(x) = 1$$

$$\lim_{x \to -\infty} \tanh(x) = -1$$

3. Identities

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\tanh^2(x) = 1 - \operatorname{sech}^2(x)$$

$$\coth^2(x) = 1 + \operatorname{csch}^2(x)$$

4. Deirvatives

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left(e^x - (-e^{-x})\right)$$
$$= \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh(x)$$

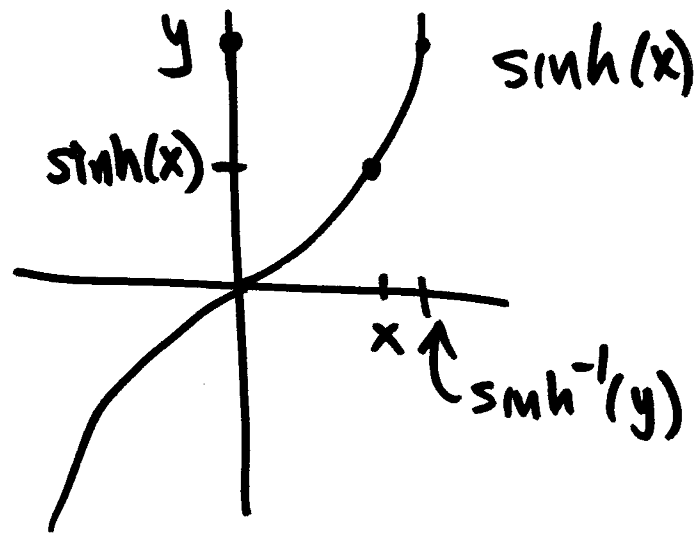
etc....

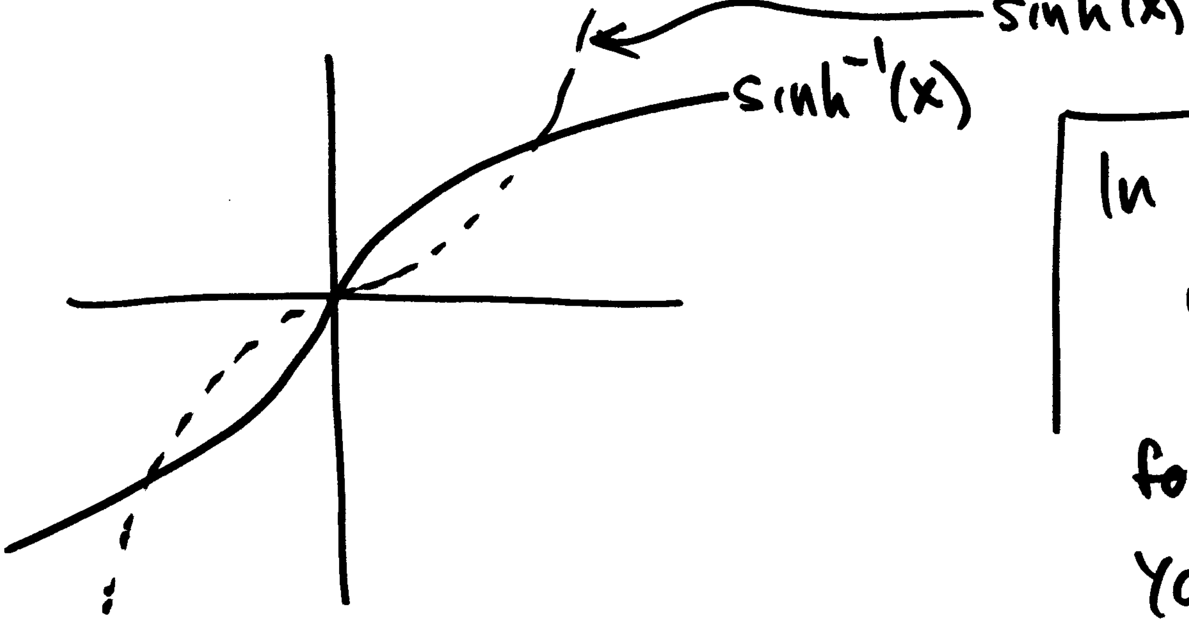
5. Inverses.

$\sinh^{-1}(x)$:

$\sinh^{-1}(x) = y \Leftrightarrow \sinh(y) = x$

for all x.





In fact you can solve

$$y = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

for x in terms of y.

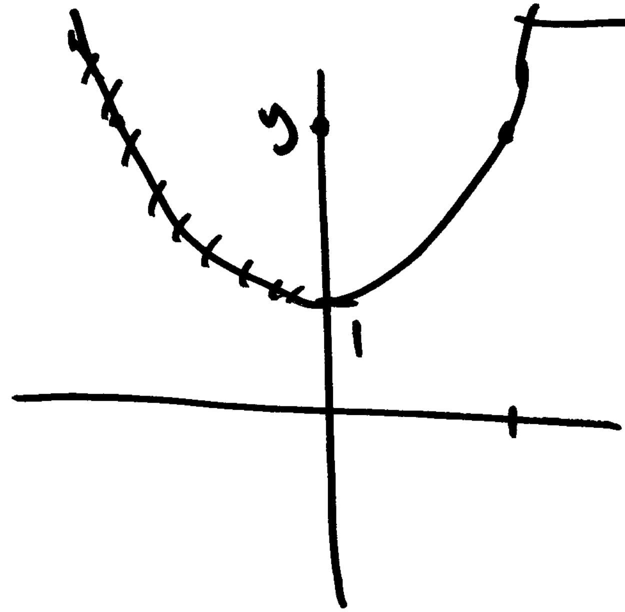
You get

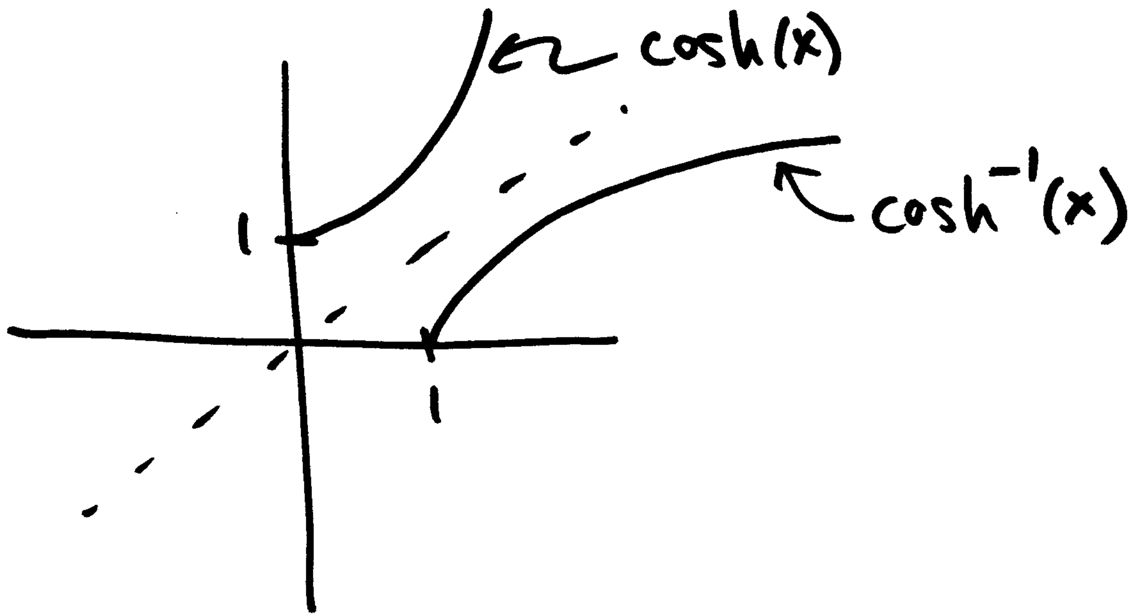
$$x = \ln\left(y + (y^2+1)^{1/2}\right)$$

$\cosh^{-1}(x)$

$\cosh^{-1}(x) = y \iff \cos(y) = x$

$x \geq 1 \qquad\qquad y \geq 0$





Derivatives

$$\frac{d}{dx}\sinh^{-1}(x)$$

$y = \sinh^{-1}(x)$ Want $\frac{dy}{dx}$.

$$\sinh(y) = x$$

$$\frac{d}{dx}\sinh(y) = \frac{d}{dx}(x)$$

$$\cosh(y)\cdot\frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\cosh(\sinh^{-1}(x))}$$

Use identity $\cosh^2(x) - \sinh^2(x) = 1$

$$\cosh^2(x) = 1 + \sinh^2(x)$$

$$\therefore \cosh^2(\sinh^{-1}(x)) = 1 + \sinh^2(\sinh^{-1}(x))$$

$$= 1 + x^2$$

$$\therefore \cosh(\sinh^{-1}(x)) = (1+x^2)^{\frac{1}{2}}$$

$$\sinh(\sinh^{-1}(x)) = x$$

$$\sinh^{-1}(\sinh(x)) = x$$

$$\therefore \frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

Similarly

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

eg #42)

$$\int \sinh\left(\frac{x}{5}\right) dx = 5\cosh\left(\frac{x}{5}\right) + c$$

Also: $\int \sinh\left(\frac{x}{5}\right) dx = \int \frac{1}{2}\left(e^{x/5} - e^{-x/5}\right) dx$

$$= \frac{1}{2}\left(5e^{x/5} - (-5)e^{-x/5}\right) + c$$

$$= \frac{5}{2}\left(e^{x/5} + e^{-x/5}\right) + c \qquad \underline{\text{ok}}$$

$$= 5\cosh\left(\frac{x}{5}\right) + c$$

#44)

$$\int 4\cosh(3x-\ln 2)\,dx$$

$$= \int 4\cdot\frac{1}{2}\left(e^{3x-\ln 2}+e^{-3x+\ln 2}\right)dx$$

$$= 2\int\left(e^{3x}\cdot \underbrace{e^{-\ln 2}}_{\frac{1}{e^{\ln 2}}=\frac{1}{2}} + e^{-3x}\cdot \underbrace{e^{\ln 2}}_{2}\right)dx$$

$$= 2\int \frac{1}{2}e^{3x}+2e^{-3x}\,dx$$

$$= 2\left(\frac{1}{2}\cdot\frac{1}{3}e^{3x}+2\cdot-\frac{1}{3}e^{-3x}\right)+C$$

$$= \frac{1}{3}e^{3x}-\frac{4}{3}e^{-3x}+C$$ OK