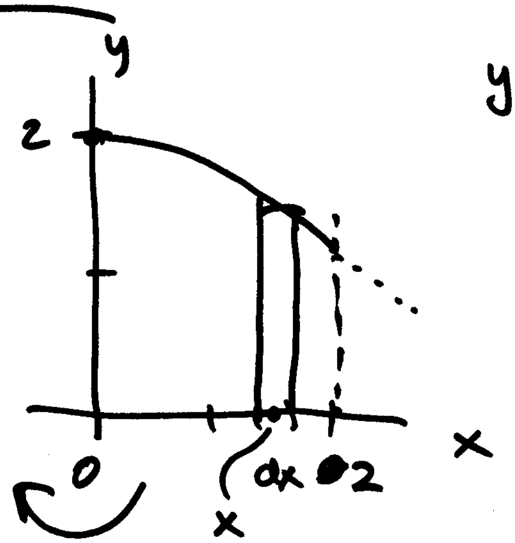
Exam 1 - Monday

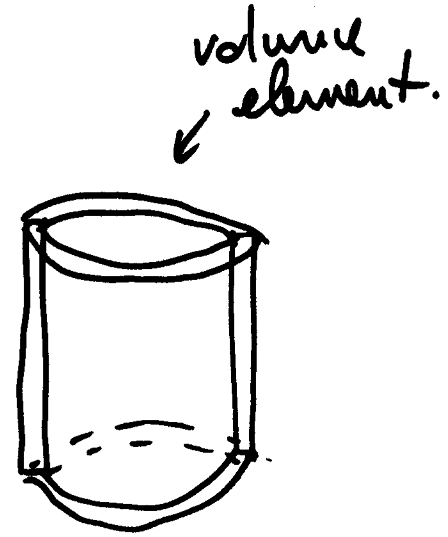
Coverage 6.1 – 6.5

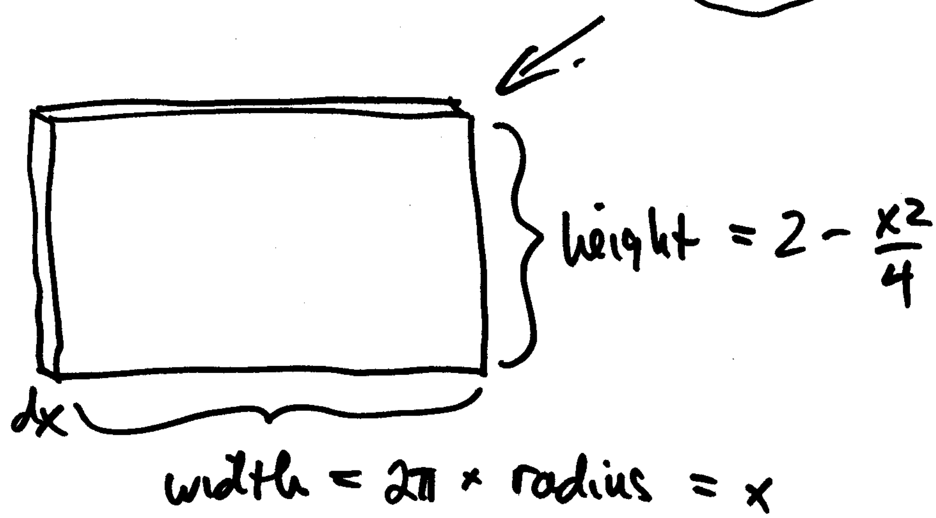
Continuing with 6.2 (cylindrical shells)

#2 p443



$y = 2 - \frac{x^2}{4}$



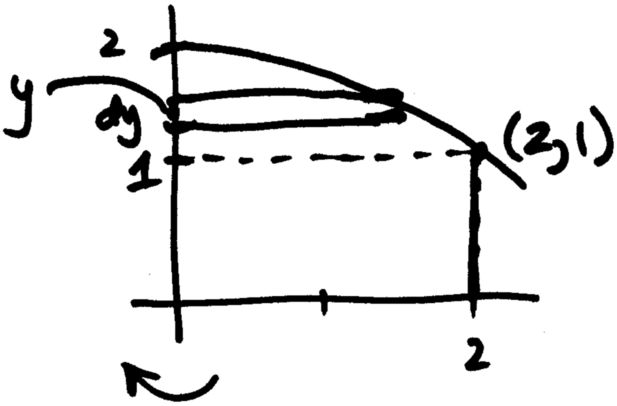


$$dV = (2\pi x)\left(2 - \frac{x^2}{4}\right) dx$$

$$V = \int_0^2 dV = \int_0^2 2\pi x\left(2 - \frac{x^2}{4}\right) dx = 2\pi \int_0^2 2x - \frac{1}{4}x^3 \, dx$$

$$= 2\pi \left(x^2 - \frac{1}{16}x^4 \Big|_0^2\right) = 2\pi(4 - 1 - 0) = 6\pi$$

What about disks/washers?



$y = 2 - \frac{1}{4}x^2$

$y - 2 = -\frac{1}{4}x^2$

$4(2-y) = x^2$

$x = 2(2-y)^{1/2}$

$$dV = \pi\left(2(2-y)^{1/2}\right)^2 dy \quad \text{for } 1 \le y \le 2$$

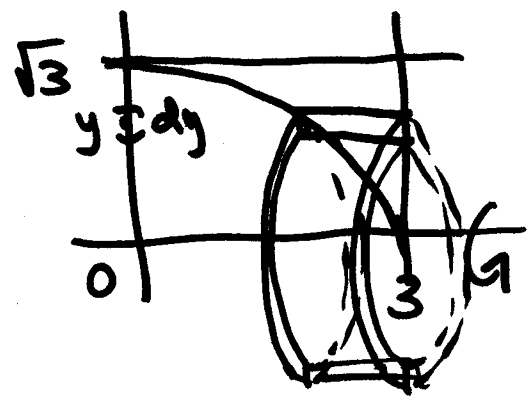
$$V = \left[\int_1^2 dV\right] + \left[\pi(2)^2 \cdot 1\right]$$

$$= \int_1^2 \pi 4(2-y)\,dy + 4\pi$$

$$= 4\pi\int_1^2 (2-y)\,dy + 4\pi = \cdots = 6\pi$$

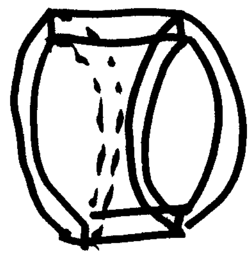
Doable but less convenient

eg #4



$x = 3 - y^2$

By shells:



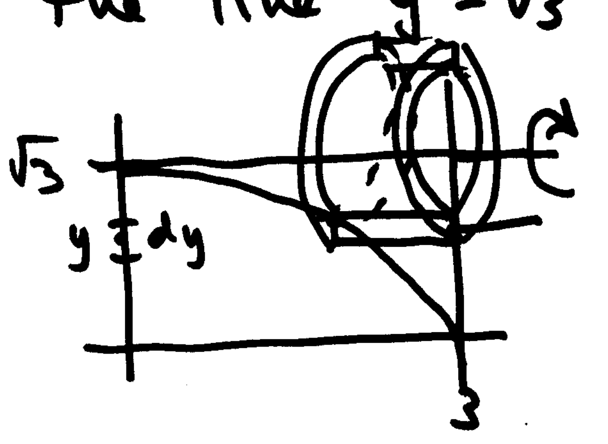
$$dV = 2\pi\,(\text{radius})(\text{height})(\text{thickness})$$

$$= 2\pi\, y\, (\underbrace{3-(3-y^2)}_{y^2})\, dy$$

$$= 2\pi y^3\, dy$$

$$V = \int_0^{\sqrt{3}} dV = \int_0^{\sqrt{3}} 2\pi y^3\, dy \quad \text{etc...}$$

Now suppose we rotate region about the line $y=\sqrt{3}$



$x = 3 - y^2$

$$dV = 2\pi\,(\text{radius})(\text{height})(\text{thickness})$$

$$= 2\pi(\sqrt{3}-y)(3-(3-y^2))\, dy$$

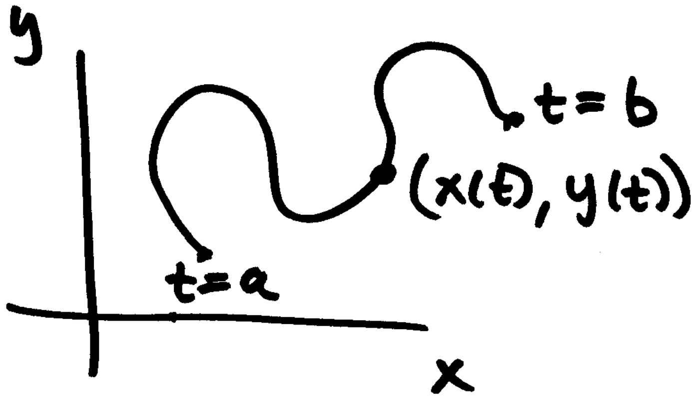
$$= 2\pi(\sqrt{3}-y)(y^2)\, dy$$

$$V = \int_0^{\sqrt{3}} dV = \int_0^{\sqrt{3}} 2\pi(\sqrt{3}-y)\,y^2\,dy \quad \text{etc...}$$

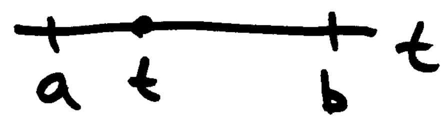
expect different answer than prev. example

6.3 Lengths of Plane curves

Problem



Find length of curve.

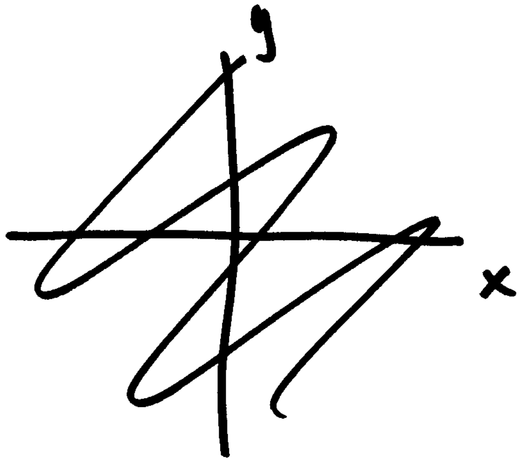


Q: How do you represent a curve in the plane?

1. As graph of function $y = f(x)$
2. As graph of function $x = g(y)$
3. As set of parametric equations.

e.g. $x = t^3 \quad y = \frac{3}{2}t^2 \qquad 0 \le t \le \sqrt{3}$

Graph:



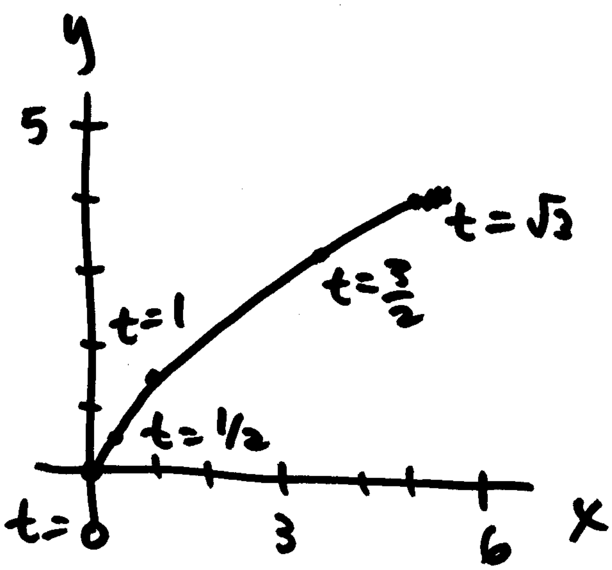
$t=0: (0,0)$

$t=1: (1, 3/2)$

$t=\frac{1}{2}: \left(\frac{1}{8}, \frac{3}{8}\right)$

$t=\frac{3}{2}: \left(\frac{27}{8}, \frac{27}{8}\right)$

$t=\sqrt{3}: \left(3\sqrt{3}, \frac{9}{2}\right)$



OR

Eliminate parameter:

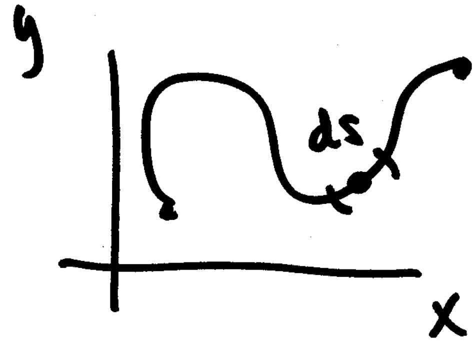
$(x)^{2/3} = (t^3)^{2/3} = t^2$

$x^{2/3} = t^2$

$y = \frac{3}{2}t^2$

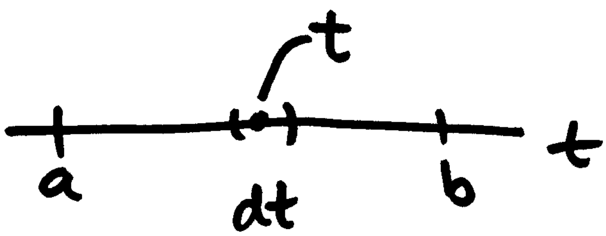
$y = \frac{3}{2}x^{2/3}$

Idea:

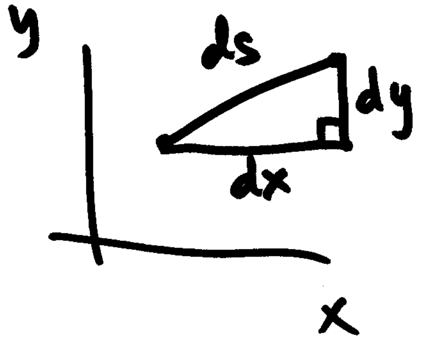


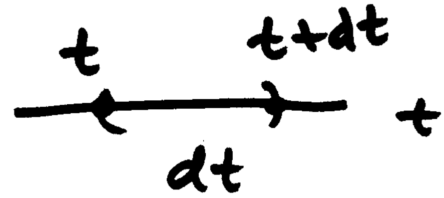
$x = x(t)$
$y = y(t)$
$a \le t \le b$

Find length



ds = arclength element.





$x = x(t) \qquad y = y(t)$

$dx = x(t+dt) - x(t) = x'(t)\,dt = \frac{dx}{dt}\cdot dt$

$dy = y(t+dt) - y(t) = y'(t)\,dt = \frac{dy}{dt}\cdot dt$

$ds^2 = dx^2 + dy^2$

$ds = (dx^2 + dy^2)^{1/2}$

$= \left(\left[\frac{dx}{dt}\right]^2 \cdot dt^2 + \left[\frac{dy}{dt}\right]^2 \cdot dt^2\right)^{1/2}$

$= \left(\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right)^{1/2} dt$ /

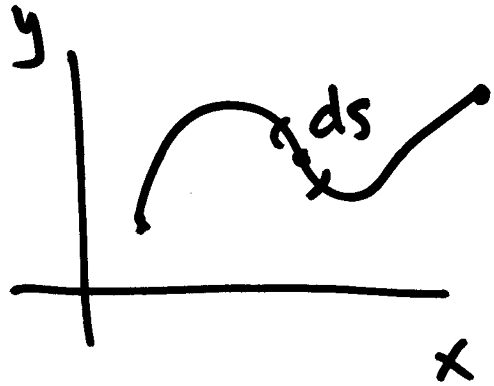
$= \left(x'(t)^2 + y'(t)^2\right)^{1/2} dt.$ /

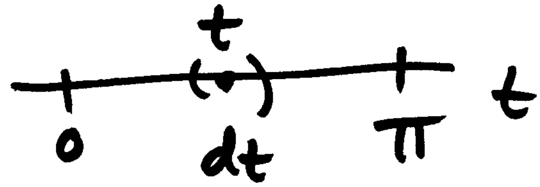
e.g. #2 p452

$x = \cos(t)$

$y = t + \sin(t)$

$0 \le t \le \pi$





$$ds = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right)^{1/2} dt$$

$$\frac{dx}{dt} = -\sin(t) \qquad \frac{dy}{dt} = (1 + \cos(t))$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2(t) + (1 + \cos(t))^2$$

$$= \sin^2(t) + 1 + 2\cos(t) + \cos^2(t)$$

$$= 2 + 2\cos(t)$$

$$ds = (2 + 2\cos(t))^{1/2} dt$$

$$s = \int_0^{\pi} ds = \int_0^{\pi} (2 + 2\cos(t))^{1/2} dt$$

$$= \sqrt{2} \int_0^{\pi} (1 + \cos(t))^{1/2} \, dt$$

$$\boxed{\begin{aligned} \sin^2\theta &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) \\ \cos^2\theta &= \tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta) \end{aligned}}$$

$$= \sqrt{2} \int_0^{\pi} \Big(2 \underbrace{\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(t)\right)}_{\cos^2\left(\frac{1}{2}t\right)}\Big)^{1/2} dt$$

$$= 2 \int_0^{\pi} \left(\cos^2\left(\tfrac{1}{2}t\right)\right)^{1/2} dt$$

$$= 2 \int_0^{\pi} \cos\left(\tfrac{1}{2}t\right) dt \quad \text{etc} \ldots$$

eg #4

$x = \frac{t^2}{2}$

$y = \frac{(2t+1)^{3/2}}{3}$

$0 \le t \le 4$

Find arclength.

$ds = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right)^{1/2} dt$

$\frac{dx}{dt} = t \quad \frac{dy}{dt} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} (2t+1)^{1/2} \cdot \cancel{2}$

$= (2t+1)^{1/2}$

$ds = (t^2 + 2t + 1)^{1/2} dt$

$= ((t+1)^2)^{1/2} dt$

$= (t+1)\, dt$

$S = \int_0^4 ds = \int_0^4 (t+1)\, dt = \text{etc} \ldots$

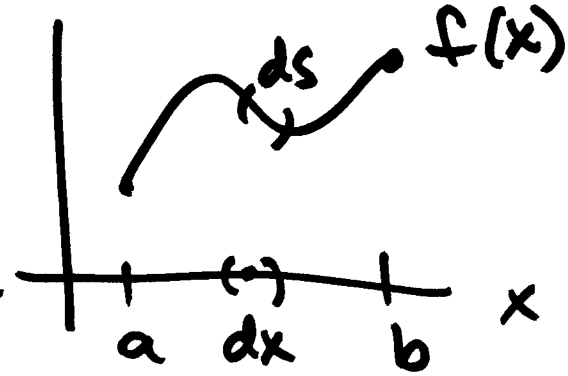
Note that:

$x = x(t) \quad y = y(t) \quad a \le t \le b$ (parametrized)

We are OK with this.

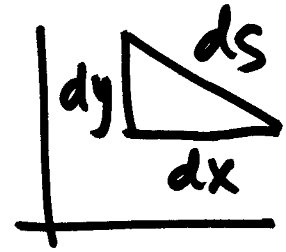
What if given $y = f(x)$, $a \le x \le b$?

$$ds = (dx^2 + dy^2)^{1/2}$$



$$= \left(dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2\right)^{1/2}$$

$$= \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$



$$= \left(1 + f'(x)^2\right)^{1/2} dx$$

$$s = \int_a^b ds = \underline{\int_a^b \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx}$$

$$dy = f'(x)\, dx$$
$$= \left(\frac{dy}{dx}\right) dx$$

what about $x = g(y)$ $\quad c \leq y \leq d$?

$$ds = (dx^2 + dy^2)^{1/2}$$

$$\frac{dx}{dy} = g'(y)$$

$$= \left(\left[\frac{dx}{dy}\right]^2 dy^2 + dy^2\right)^{1/2}$$

$$S = \int_c^d ds$$

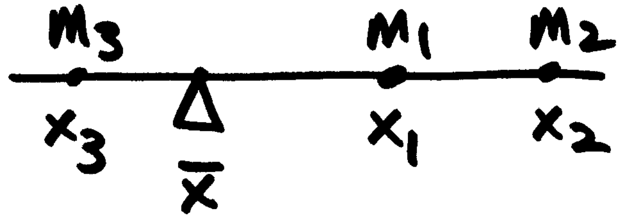
$$= \left(\left[\frac{dx}{dy}\right]^2 + 1\right)^{1/2} dy$$

$$= \left(g'(y)^2 + 1\right)^{1/2} dy$$

$$= \underline{\int_c^d \left(g'(y)^2 + 1\right)^{1/2} dy}$$

6.4 Moments + Centers of Mass

Idea: Example



For balance:

$$m_1(x_1-\bar{x})+m_2(x_2-\bar{x})=m_3(\bar{x}-x_3)$$

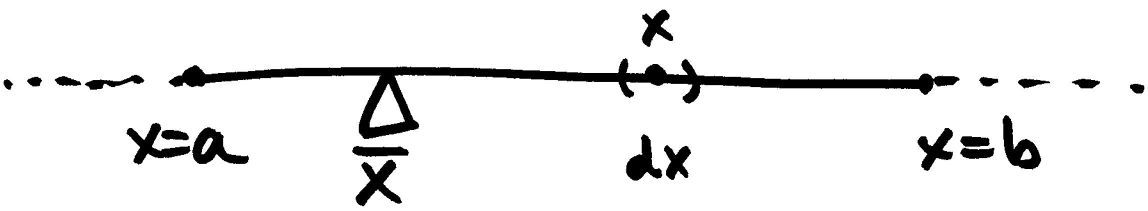
Solve for $\bar{x}$

$$\underbrace{m_1x_1+m_2x_2+m_3x_3}_{\substack{\text{moment}\\ \text{(about the origin)}\\ \text{of system}}} = \bar{x}\,\underbrace{(m_1+m_2+m_3)}_{\substack{\text{total mass}\\ \text{of}\\ \text{system}}}$$

$$\bar{x}=\frac{\text{moment of system}}{\text{mass of system}}=\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}$$

$\bar{x}$ = center of mass.

Calculus required when you have a solid bar or wire.



1. To make problem non-trivial we assume non-constant density $\delta(x)$ (measured in $\frac{\text{mass}}{\text{length}}$)

2. Think of the wire or bar as a collection of tiny sub-rods acting like point masses.

dm = mass element of small piece of rod

$= \delta(x)\,dx$

moment about origin of small piece

$= x\,dm = x\,\delta(x)\,dx$

moment about origin of whole rod

$= M_0 = \int_a^b x\,dm = \int_a^b x\,\delta(x)\,dx$

Total mass of rod =

$$M = \int_a^b dm = \int_a^b \delta(x)\,dx$$

Center of mass

$$\overline{x} = \frac{M_0}{M} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx}$$

<u>eg</u> #10

$$\delta(x) = 3\left(x^{-3/2} + x^{-5/2}\right), \quad \frac{1}{4} \le x \le 1$$

Find M_0, M, $\overline{x}$

moment about origin

total mass

center of mass

$$M_0 = \int_a^b x\,\delta(x)\,dx = \int_{\frac{1}{4}}^{1} 3x\left(x^{-3/2} + x^{-5/2}\right)dx$$

$$= 3\int_{\frac{1}{4}}^{1} \left(x^{-1/2} + x^{-3/2}\right)dx$$

$$= 3\left(2x^{1/2} - 2x^{-1/2}\Big|_{\frac{1}{4}}^{1}\right)$$

$$= 6\left(\overbrace{(1-1)}^{0} - \left(\tfrac{1}{2} - 2\right)\right)$$

$$= 6\left(-\tfrac{1}{2} + 2\right) = 9 \,/\!/$$

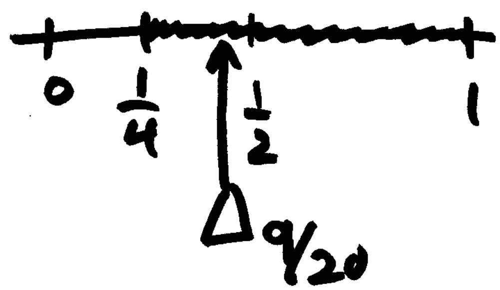
$$M = \int_{\frac{1}{4}}^{1} \delta(x)\,dx = 3\int_{\frac{1}{4}}^{1} x^{-3/2} + x^{-5/2}\,dx$$

$$= 3\left(-2x^{-1/2} - \tfrac{2}{3}x^{-3/2}\Big|_{\frac{1}{4}}^{1}\right) \qquad \left(\tfrac{1}{4}\right)^{-3/2} = 4^{3/2} = 2^{3} = 8$$

$$= 3\left(-2 - \tfrac{2}{3} - \left(-4 - \tfrac{2}{3}\cdot 8\right)\right)$$

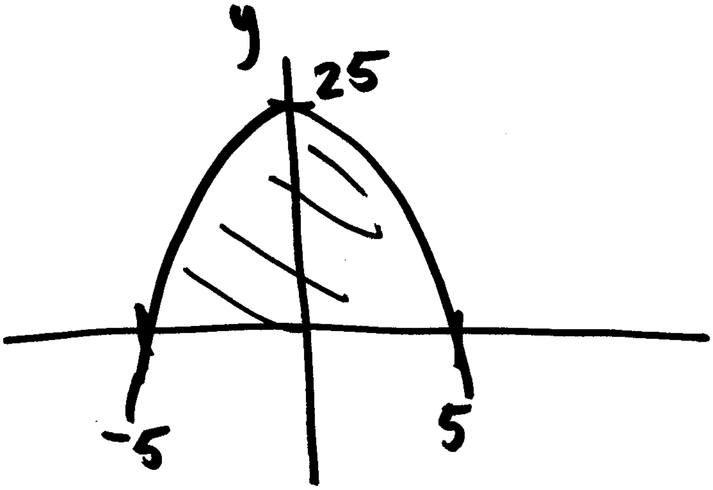
$$= 3\left(2 + \tfrac{14}{3}\right) = 20 \,/\!/$$

$$\bar{x} = \frac{M_0}{M} = \frac{9}{20}$$



e.g. #14

const density δ



$y = 25 - x^2$

Need M_x = moment about x-axis

M_y = moment about y-axis

M = total mass.

center of mass $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$

$\bar{x} = 0$ by symmetry.

M = (density) (area of plate)

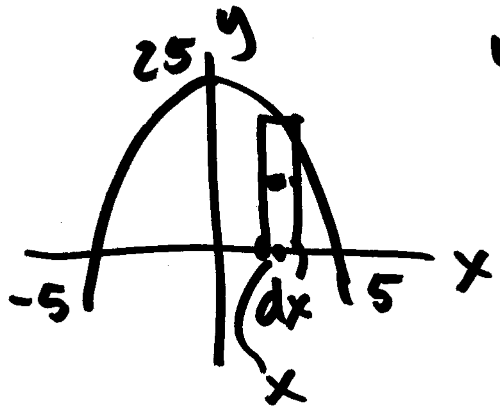
$$= \delta \int_{-5}^{5} (25 - x^2)\, dx$$

Want M_x:

Two ways

① vertical rectangles

$y = 25 - x^2$



$dm = \delta$ (area of rect.)

$= \delta\,(25 - x^2)\,dx$

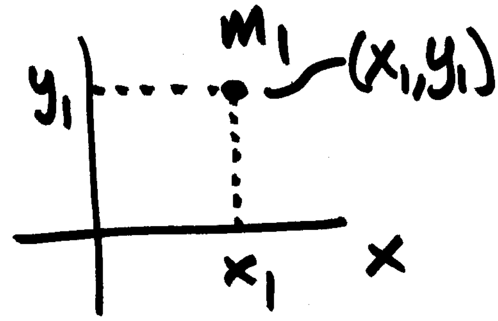
moment of mass element

$= \left(\frac{25 - x^2}{2}\right) dm$

$$M_x = \int_{-5}^{5} \frac{25 - x^2}{2}\, dm$$

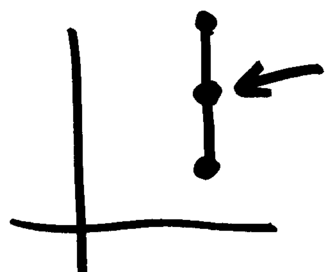
$$= \delta \int_{-5}^{5} \frac{1}{2}(25 - x^2)^2\, dx$$

eg point mass



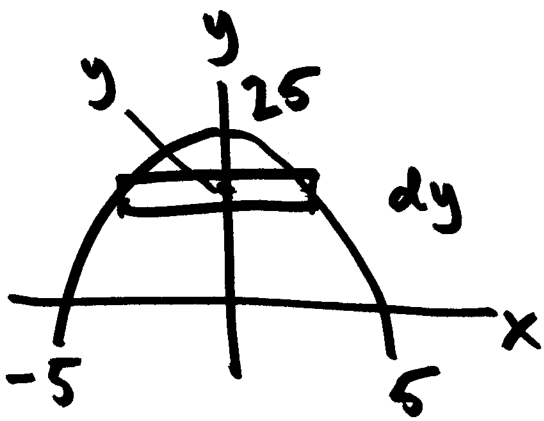
moment about x-axis is $m_1 y_1$

e.g. rod w/const density



Can think of rod as a point mass located at center of mass which is at midpoint.

② horizontal rectangles



$y = 25 - x^2$

$x^2 = 25 - y$

$x = (25-y)^{1/2}$

$dm = \delta \cdot (\text{area of rect.})$

$= \delta \cdot 2(25-y)^{1/2}\, dy$

moment element $= y\, dm$

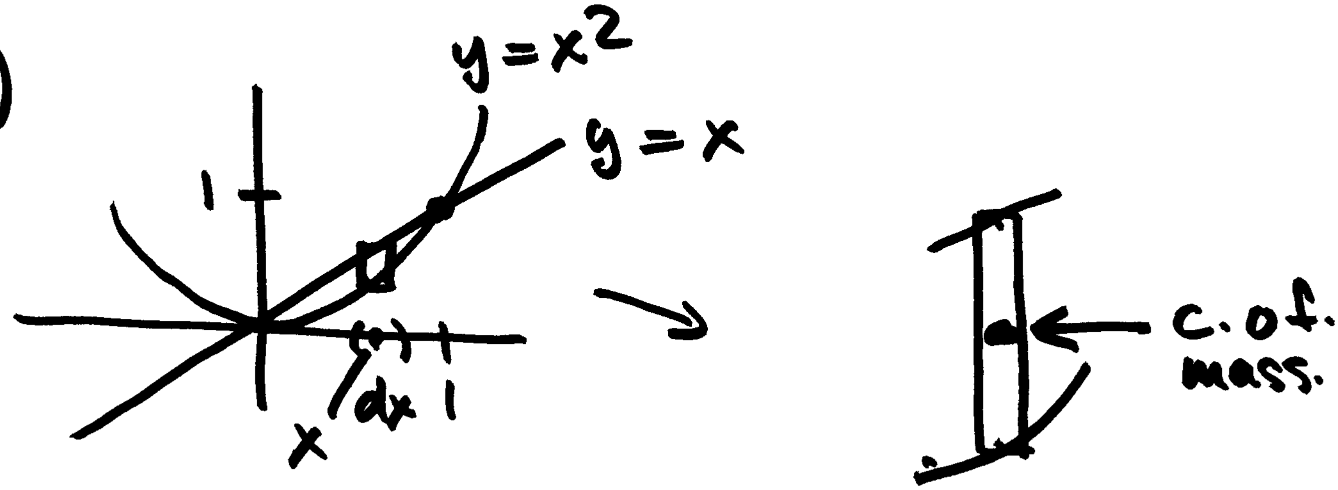
$$M_x = \int_0^{25} y\, dm = \int_0^{25} 2\delta\, y (25-y)^{1/2}\, dy$$

$$M = \int_0^{25} dm = 2\delta \int_0^{25} (25-y)^{1/2}\, dy$$

$$\bar{y} = \frac{M_x}{M}$$

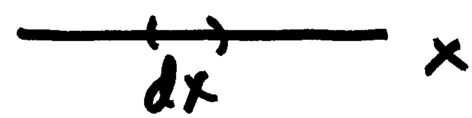
Variable densities

#28)



Find M_x:

$\delta(x,y)=12x$



mass element

$$dm = \underbrace{12x}_{\text{density}}(x-x^2)\,dx = 12x(x-x^2)\,dx$$

$$\text{moment element} = \left(\frac{x+x^2}{2}\right)dm$$

$$M_x = \int_0^1 \left(\frac{x+x^2}{2}\right)dm = \int_0^1 \left(\frac{x+x^2}{2}\right)12x(x-x^2)\,dx$$