Review of Chapter 5

#83) #79

79) $\displaystyle\int_0^1 \frac{36\,dx}{(2x+1)^3}$

$u = 2x+1$

$du = 2\,dx$ $\quad \left[\ \frac{du}{dx} = 2 \quad du = 2\,dx\ \right]$

$$= \int_0^1 \frac{18 \cdot \overbrace{2\,dx}^{du}}{\underbrace{(2x+1)^3}_{u^3}}$$

$x = 0 \quad u = 1$

$x = 1 \quad u = 3$

$$= 18\int_1^3 \frac{du}{u^3} = 18\int_1^3 u^{-3}\,du$$

$$= 18 \cdot \frac{-1}{2}u^{-2}\Big|_1^3 = -9u^{-2}\Big|_1^3$$

$$= -9\left(3^{-2} - 1^{-2}\right) = -9\left(\frac{1}{9} - 1\right) = 8 \,/\!/$$

83) $\displaystyle\int_0^{\pi} \sin^2(5r)\,dr$

$$= \int_0^{\pi}\left[\frac{1}{2} - \frac{1}{2}\cos(10r)\right]dr$$

$$= \frac{1}{2}r - \frac{1}{2}\cdot\frac{1}{10}\sin(10r)\Big|_0^{\pi}$$

$$= \frac{1}{2}r - \frac{1}{20}\sin(10r)\Big|_0^{\pi}$$

Trig identity:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

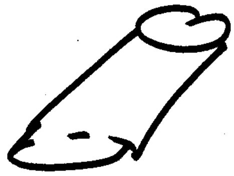
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

$$= \left(\tfrac{1}{2}\pi - \underbrace{\cancel{\tfrac{1}{20}\sin(10\pi)}}_{0}\right) - \left(\tfrac{1}{2}\cdot 0 - \underbrace{\cancel{\tfrac{1}{20}\sin(0)}}_{0}\right) = \frac{\pi}{2}$$ //

Continuing 6.1 :

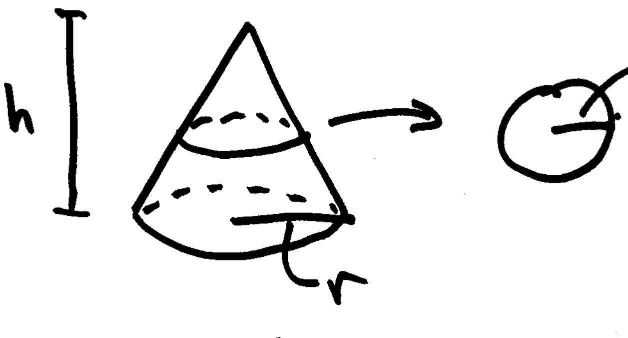
Idea: Volumes by slicing

$$V = \underbrace{\pi r^2}_{\text{area of cross-section}} \cdot \underbrace{h}_{\text{height}}$$



If cross-section areas not constant then calculus required.

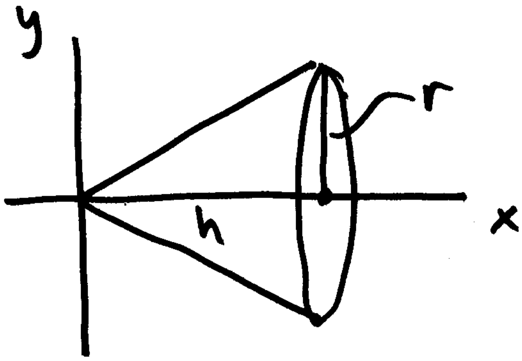
Example: Volume of a cone.

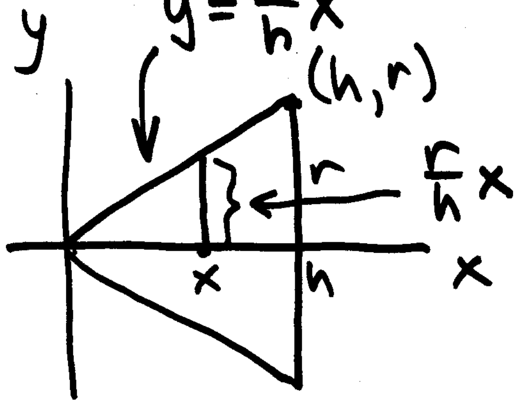


radius depends on location of cross-section

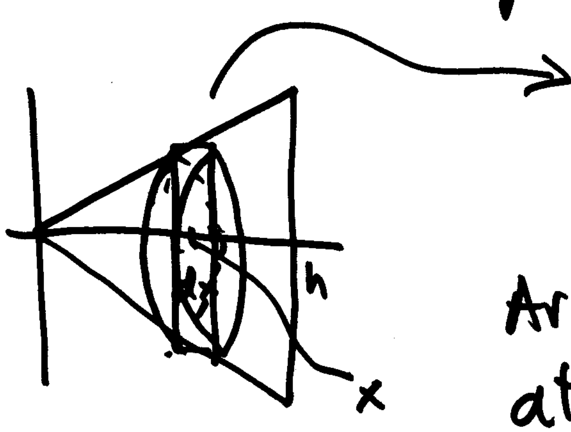
Idea: Divide cone into small (infinitesimal) volume elements. Then add up elements. using integral.

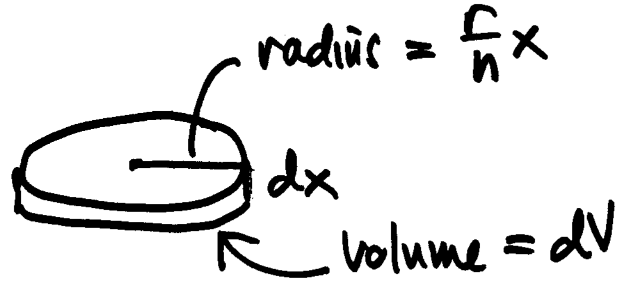
Set up:





Each volume element is a tiny disk obtained by:





Area of cross-section at x $= A(x) = \pi\left(\frac{r}{h}x\right)^2$

$$= \pi \frac{r^2}{h^2} x^2$$

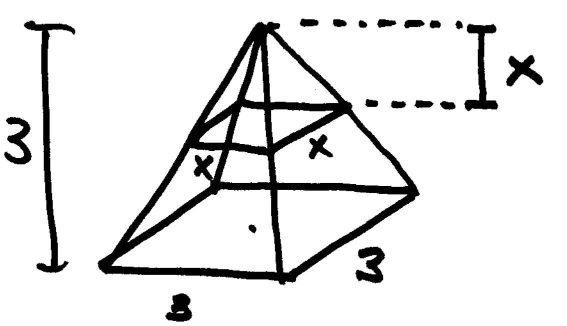
Volume element at x $= dV = A(x)\,dx = \frac{\pi r^2}{h^2} x^2\,dx$

Volume of cone $= \displaystyle\int_0^h dV = \int_0^h \frac{\pi r^2}{h^2} x^2\,dx$

$$= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi r^2}{h^2}\cdot \frac{1}{3}x^3\Big|_0^h = \frac{\pi r^2}{h^2}\left(\frac{1}{3}h^3 - \frac{1}{3}0^3\right)$$

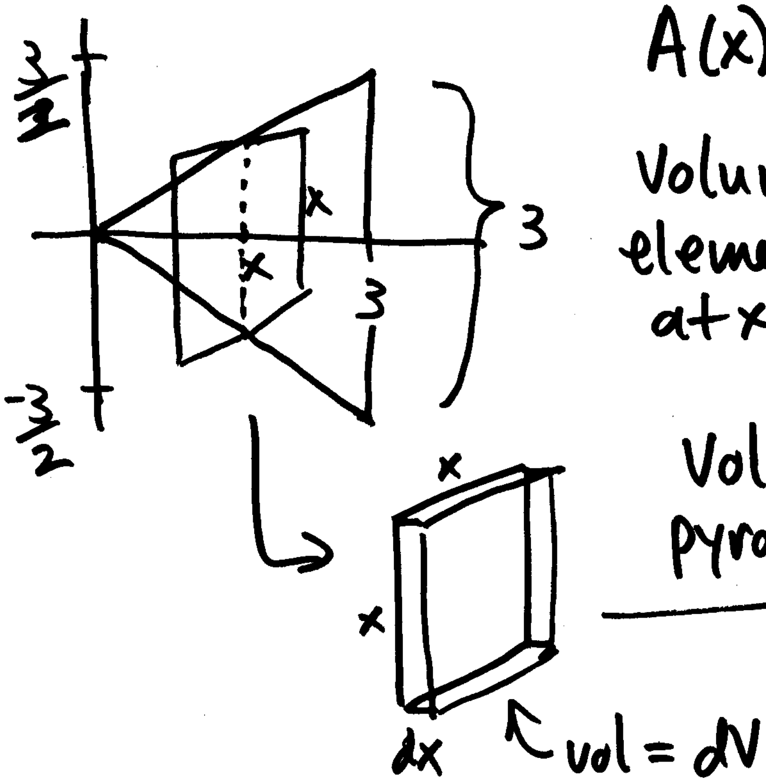
$$= \frac{\pi r^2}{h^2}\cdot \frac{1}{3}h^3 = \frac{1}{3}\pi r^2 h.$$

Example 1 p427



Want volume of pyramid.

Need: ~~Fo~~ Formula for cross-sectional area at x.



$$A(x) = x^2$$

Volume element at x $= dV = x^2\,dx$

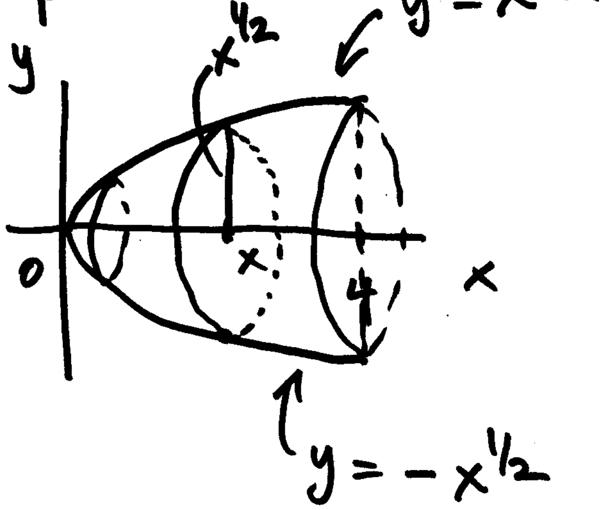
Vol of pyramid $= \int_0^3 dV = \int_0^3 x^2\,dx$

$$= \frac{1}{3}x^3\Big|_0^3 = \frac{1}{3}(27-0)$$

$$= 9 \text{ m}^3.$$

#2 p 434

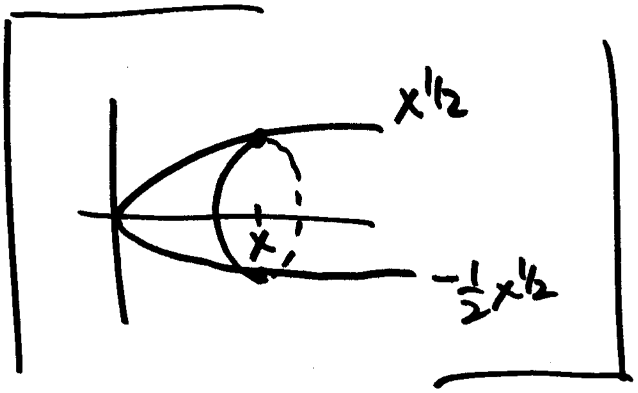
a)



$A(x) =$ cross-section area at x

$= \pi (x^{1/2})^2 = \pi x$

Vol element at x $= dV = \pi x \, dx$

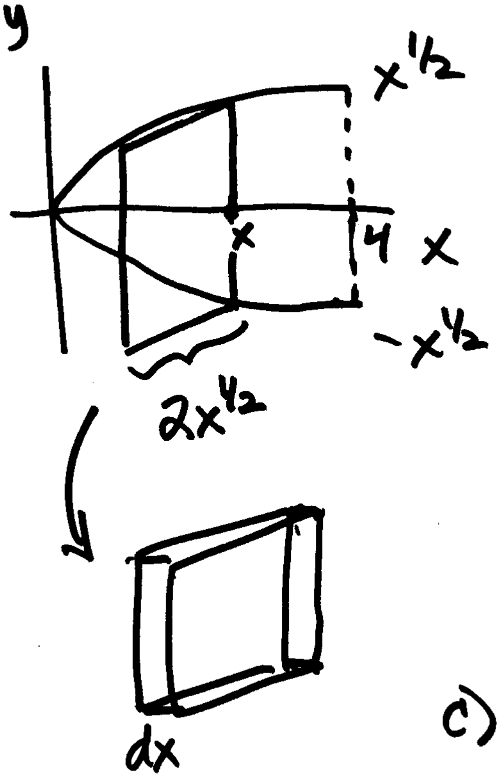


$$\text{Vol} = \int_0^4 dV = \int_0^4 \pi x \, dx$$

$$= \pi \frac{1}{2} x^2 \Big|_0^4 = \frac{\pi}{2}(16-0)$$

$$= 8\pi \,//.$$

b)

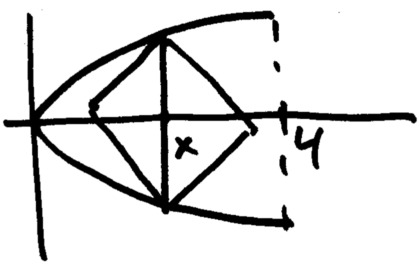


$$A(x) = (2x^{1/2})^2 = 4x$$

$$dV = 4x \, dx$$

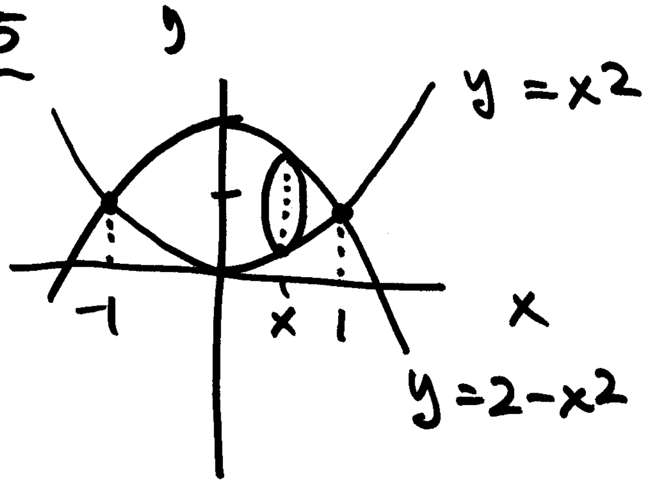
$$V = \int_0^4 dV = \int_0^4 4x \, dx = 32$$

c)



etc...

#4 p 435



$x^2 = 2 - x^2$

$2x^2 = 2$

$x^2 = 1 \quad \boxed{x = \pm 1}$

$$A(x) = \pi\left(\frac{\text{diameter at } x}{2}\right)^2$$

$$= \pi\left(\frac{2 - x^2 - x^2}{2}\right)^2 = \pi\left(\frac{2 - 2x^2}{2}\right)^2 = \pi(1 - x^2)^2$$

$$dV = \pi(1 - x^2)^2\,dx$$

$$V = \int_{-1}^{1} dV = \int_{-1}^{1} \pi(1 - x^2)^2\,dx$$

$u = 1 - x^2$

$du = -2x\,dx$

$x = (1 - u)^{1/2}$

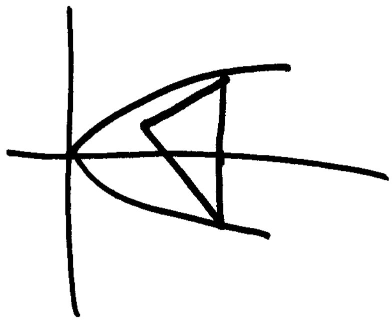
too complicated...

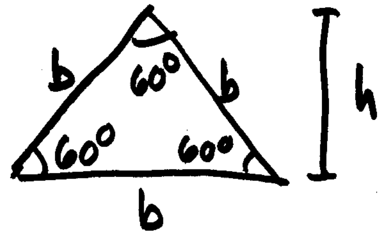
$$= \pi\int_{-1}^{1}(1 - 2x^2 + x^4)\,dx$$

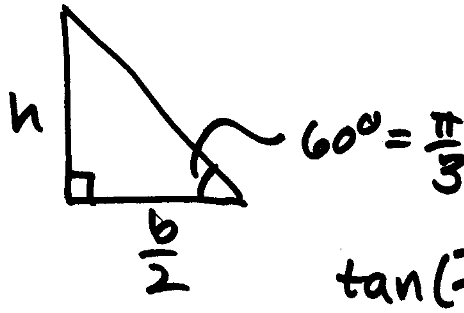
$$= \pi\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \Big|_{-1}^{1}\right)$$

$$= \pi\left(\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right)$$

$$= \pi\left(1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right) = \pi\left(2 - \frac{4}{3} + \frac{2}{5}\right) = \frac{16\pi}{15}$$







$$\tan\left(\frac{\pi}{3}\right) = \frac{h}{\frac{b}{2}} = \frac{2h}{b}$$

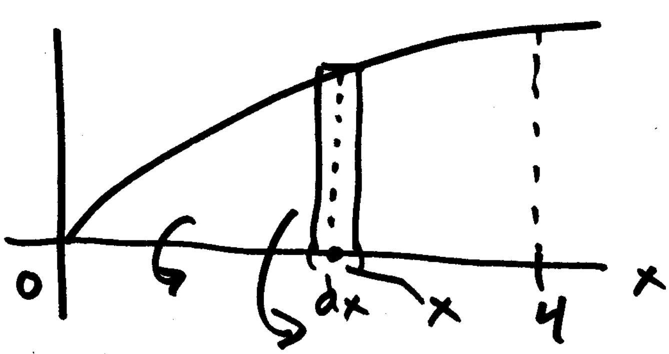
$$\parallel$$

$$\frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

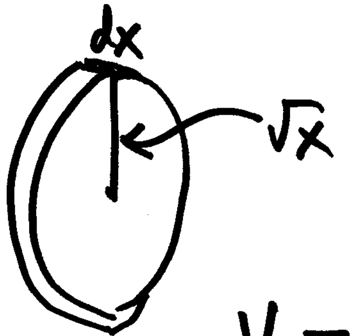
$$\underline{h = b \cdot \frac{\sqrt{3}}{2}}$$

Solids of Revolution.

<u>Example 4, p429</u>



$y = \sqrt{x}$

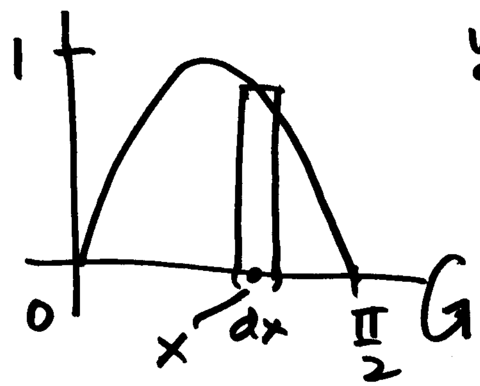


$$dV = \pi(\sqrt{x})^2\,dx$$

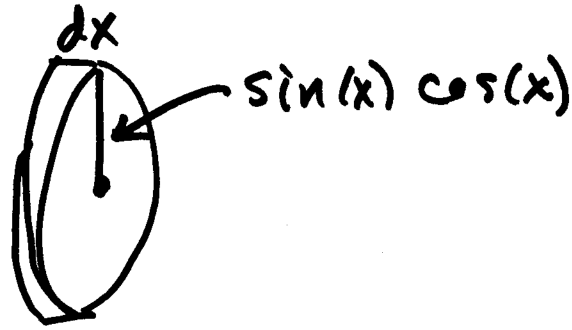
$$= \pi x\,dx$$

$$V = \int_0^4 dV = \int_0^4 \pi x\,dx = \text{etc...}$$

eg #18 p 436



$y = \sin(x)\cdot\cos(x)$



$$dV = \pi\,(\text{radius})^2\,\underbrace{dx}_{\text{thickness}}$$

$$= \pi\left(\sin(x)\cos(x)\right)^2 dx$$

$$V = \int_0^{\pi/2} dV = \int_0^{\pi/2} \pi \sin^2(x)\cos^2(x)\,dx$$

$\cos^2(x) = 1 - \sin^2(x)$

$$\int_0^{\pi/2} \pi \sin^2(x)\left(1-\sin^2(x)\right)dx$$

$u = \sin^2(x)$

$du = 2\sin(x)\cos(x)\,dx$

no help.

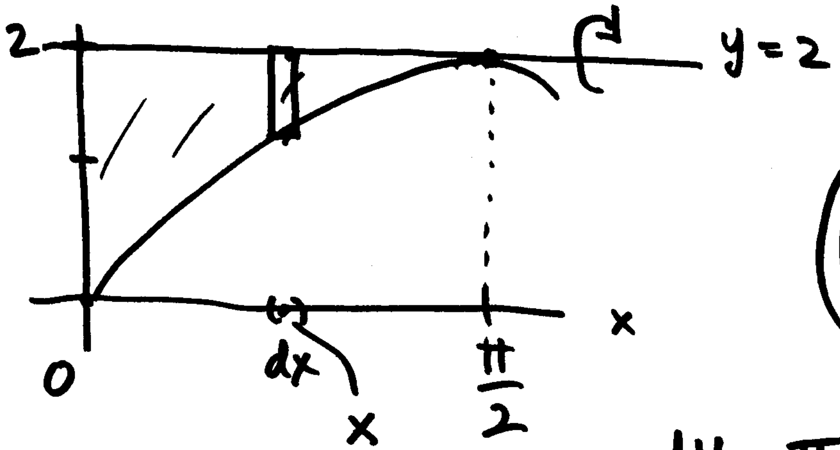
$$= \int_0^{\pi/2} \pi \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$$

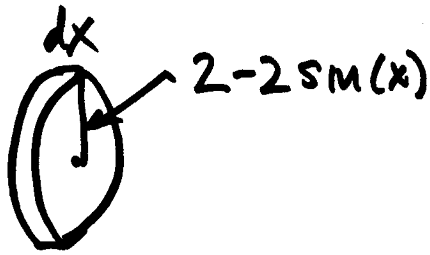
$$= \int_0^{\pi/2} \pi \left(\frac{1}{4} - \frac{1}{4}\cos^2(2x)\right) dx$$

$$= \int_0^{\pi/2} \pi \left(\frac{1}{4} - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \pi \left(\frac{1}{8} - \frac{1}{8}\cos(4x)\right) dx$$ ok from here.

e.g. #30 $\quad y = 2 \quad y = 2\sin(x) \quad 0 \le x \le \frac{\pi}{2}$

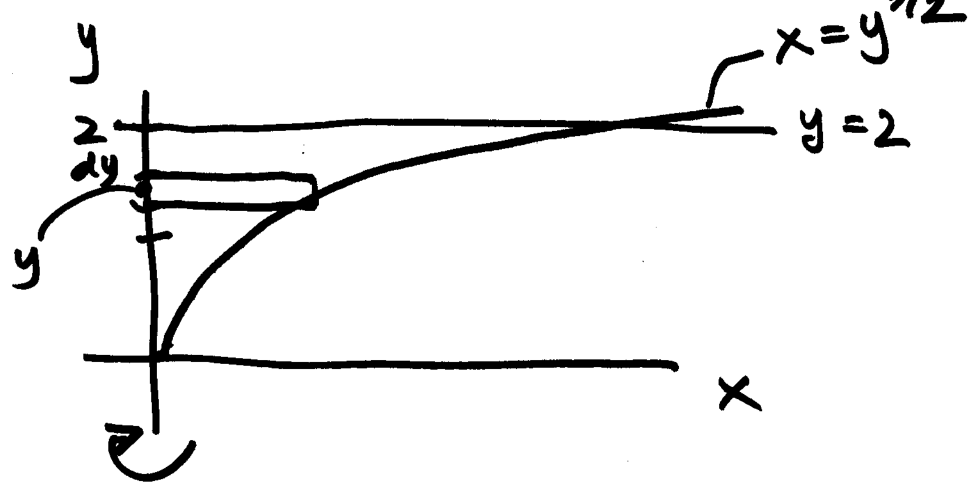




$$dV = \pi(2 - 2\sin(x))^2\, dx$$
$$= 4\pi(1 - \sin(x))^2\, dx$$

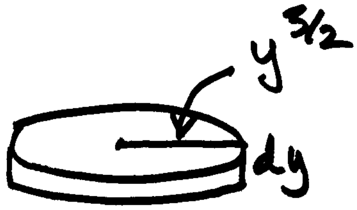
$$V = \int_0^{\pi/2} dV = \int_0^{\pi/2} 4\pi(1 - \sin(x))^2\, dx$$
$$= 4\pi \int_0^{\pi/2} 1 - 2\sin(x) + \sin^2(x)\, dx \quad \text{etc...}$$

#32) $x = y^{3/2}$, $x = 0$, $y = 2$



$x = y^{3/2}$

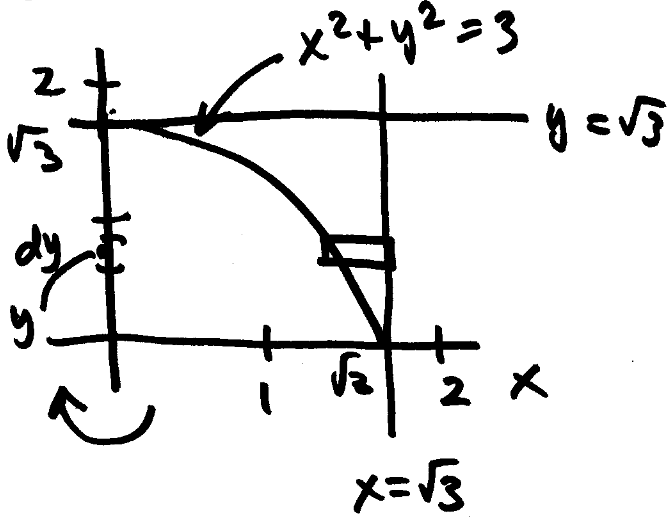
$y = x^{2/3}$



$$dV = \pi (y^{3/2})^2 \, dy = \pi y^3 \, dy$$

$$V = \int_0^2 dV = \int_0^2 \pi y^3 \, dy \text{ etc...}$$

#48)



~~$y=\sqrt{3}$~~ $x = \sqrt{3 - y^2}$

$$dV = \pi \left(\begin{smallmatrix}\text{outer}\\ \text{radius}\end{smallmatrix}\right)^2 dy - \pi \left(\begin{smallmatrix}\text{inner}\\ \text{radius}\end{smallmatrix}\right)^2 dy$$

$$= \pi \left(\left(\begin{smallmatrix}\text{outer}\\ \text{radius}\end{smallmatrix}\right)^2 - \left(\begin{smallmatrix}\text{inner}\\ \text{radius}\end{smallmatrix}\right)^2 \right) dy$$

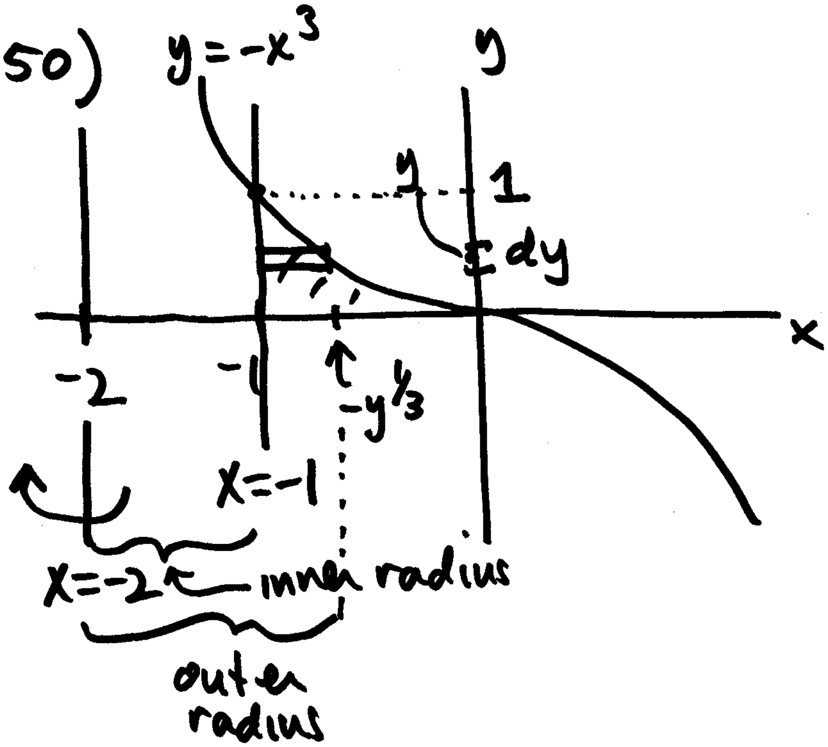
$$= \pi \left((\sqrt{3})^2 - (\sqrt{3-y^2})^2 \right) dy$$

$$= \pi \left(3 - (3 - y^2) \right) dy$$

$$= \pi y^2 \, dy$$

$$V = \int_0^{\sqrt{3}} dV = \int_0^{\sqrt{3}} \pi y^2 \, dy \quad \text{etc} \cdots$$

#50)



$$y = -x^3$$

$$x = (-y)^{1/3} = -y^{1/3}$$

 dy

$$dV = \pi\Big(\underbrace{(-y^{1/3}-(-2))^2}_{\text{outer radius}} - \underbrace{1^2}_{\text{inner radius}}\Big)\,dy$$

$$= \pi\left((2-y^{1/3})^2 - 1\right)dy$$

$$V = \int_0^1 dV = \int_0^1 \pi\left(4-4y^{1/3}+y^{2/3}-1\right)dy$$

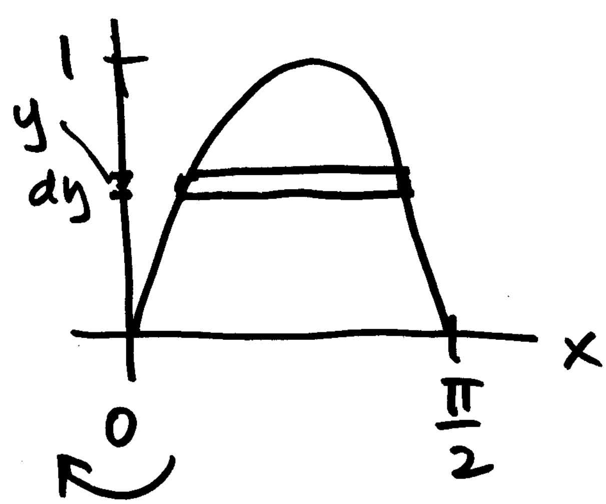
$$= \pi\int_0^1 \left(3-4y^{1/3}+y^{2/3}\right)dy = \text{etc}\ldots$$

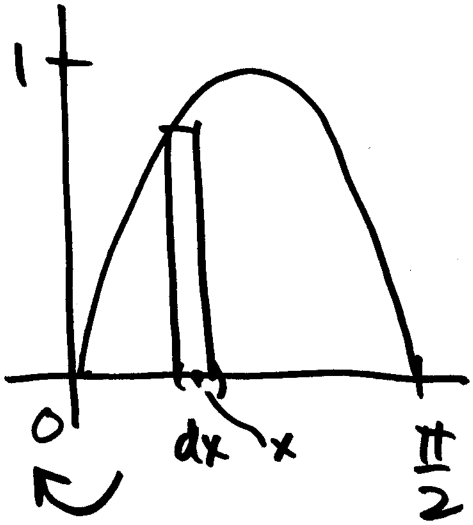
6.2 Volumes by shells.

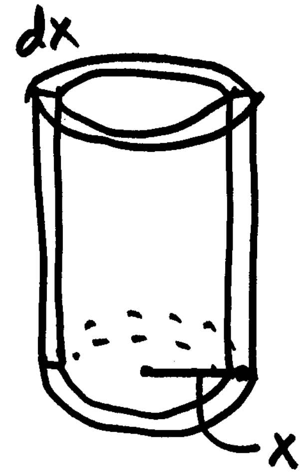
e.g. #18 p436 but rotate about y-axis

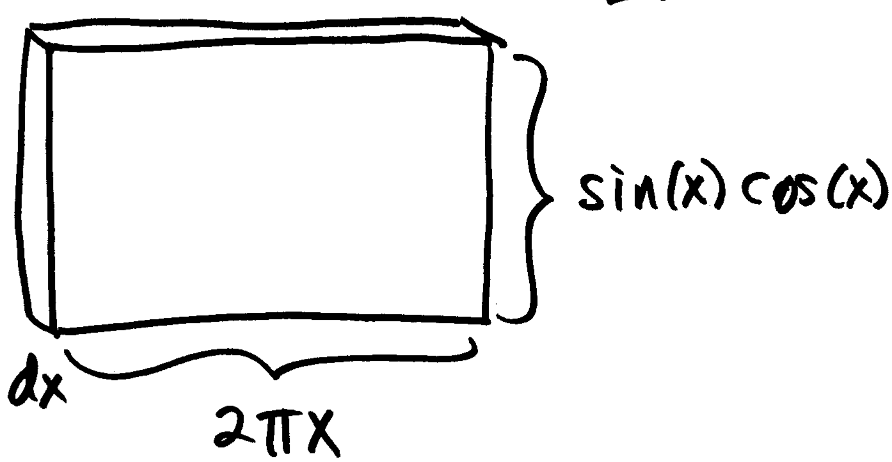
$$y = \sin(x)\cos(x)$$

Washers don't work!









$$dV = \underbrace{\begin{pmatrix}\text{circumference}\\ \text{of shell}\end{pmatrix}}_{2\pi\cdot\begin{pmatrix}\text{radius}\\ \text{of shell}\end{pmatrix}} \times \begin{pmatrix}\text{height of}\\ \text{shell}\end{pmatrix} \cdot \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi x \sin(x)\cos(x)\, dx$$

$$V = \int_0^{\pi/2} dV = \int_0^{\frac{\pi}{2}} 2\pi x \sin(x)\cos(x)\, dx$$

can't do this yet! But we will later.