MATH 114-C01

This week office hrs TR 2-3

NO CLASS WED JULY 4.

Chapter 5 Review

1. Fundamental Thm of Calculus

$$\int_a^b f(x)\,dx$$

ⓐ $F(x) = \int_a^x f(t)\,dt$ satisfies

$F'(x) = f(x)$

We say $F(x)$ is an anti-derivative of $f(x)$.

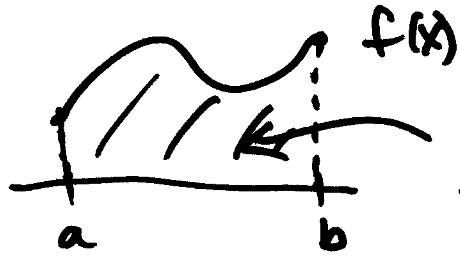
Anti-derivatives are not unique.

If $F(x)$ and $G(x)$ both satisfy $F'(x) = f(x)$ and $G'(x) = f(x)$ then $G(x) = F(x) + C$ (C = const.)

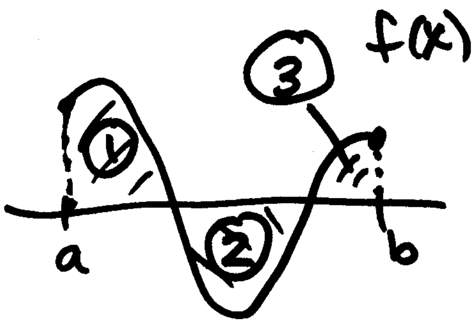
ⓑ $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F(x)$ is <u>any</u> antiderivative of $f(x)$.

Interpretation of $\int_a^b f(x)\,dx$.

$\int_a^b f(x)\,dx$ = "area" under the graph of $f(x)$ between $x=a$, $x=b$.

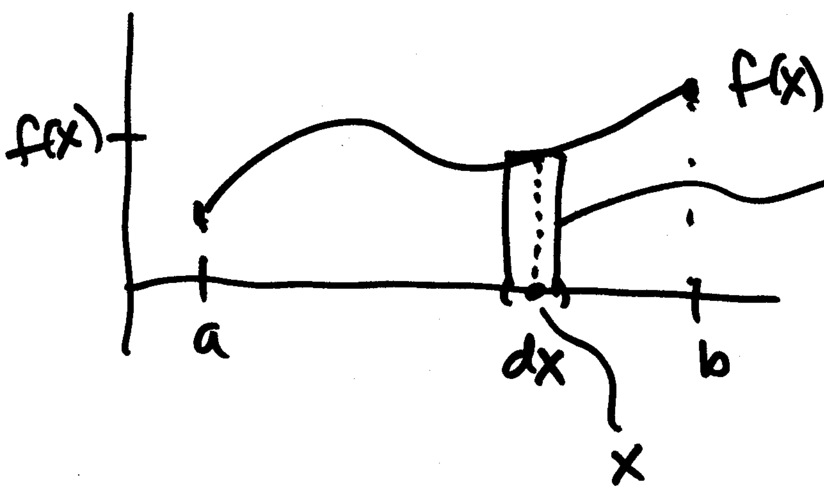


$\int_a^b f(x)\,dx$ = area



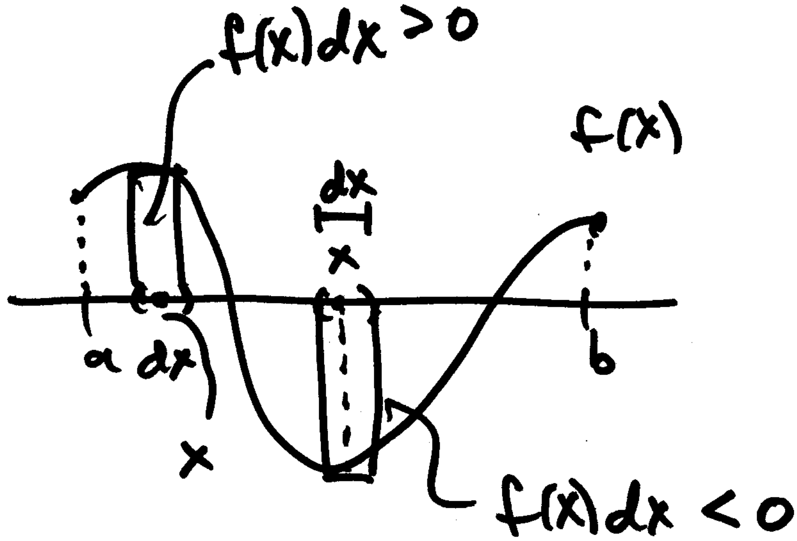
$\int_a^b f(x)\,dx$ = Area (1) + Area (3) − Area (2)

$\int_{x=a}^{x=b} \underbrace{f(x)\,dx}$ "=" sum of "areas" of Infinitesimal rectangles.



$f(x)\,dx$ = area of tiny rectangle

$f(x) \qquad f(x)\,dx$



FTC

(a) $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$

(b) $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$

(b) $\longrightarrow$ (a) $\quad \underbrace{\int_a^x f(t)\,dt}_{F(x)} = G(x) - G(a)$

where G is <u>any</u> a.d. of $f(x)$

Then $\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\left(G(x) - G(a)\right)$

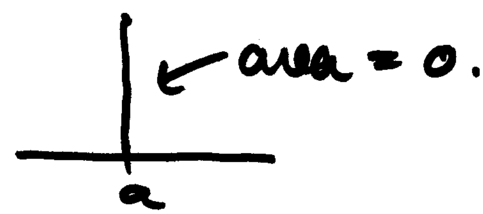
$= G'(x) - 0 = f(x)$

ⓐ → ⓑ Let $F(x)$ be <u>any</u> antiderivative of $f(x)$ then

$$\underbrace{\int_a^x f(t)\,dt}_{\substack{\text{specific}\\ \text{a.d.}}} = \underbrace{F(x)}_{\substack{\text{any}\\ \text{a.d.}}} + C$$

What is C? Let $x = a$.

$$\underbrace{\int_a^a f(t)\,dt}_{0} = F(a) + C$$



then $C = -F(a)$

$$\therefore \int_a^x f(t)\,dt = F(x) - F(a)$$

Let $x = b$ then you get part (b) of FTC.

i.e. $\int_a^b f(t)\,dt = F(b) - F(a)$

e.g. $\displaystyle\int_0^2 (x^3 - 2x^2 + 3)\,dx$

$$\boxed{\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \qquad \begin{array}{c} r=0 \\ 1 = x^0 \to \frac{1}{1}x^1 = x \end{array} \\ \frac{d}{dx}\left(\frac{1}{r+1}x^{r+1}\right) = \frac{1}{r+1}(r+1)x^{r+1-1} = x^r}$$

$$= \left(\frac{1}{4}x^4 - 2\cdot\frac{1}{3}x^3 + 3x\right)\Big|_0^2$$

$$= \left(\frac{1}{4}(2)^4 - \frac{2}{3}(2)^3 + 3(2)\right) - \left(\frac{1}{4}(0)^4 - \frac{2}{3}(0)^3 + 3\cdot 0\right)$$

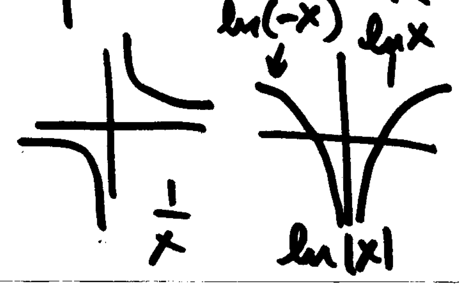
$$= 4 - \frac{16}{3} + 6 = \frac{14}{3}$$

e.g. $\displaystyle\int_1^2 \left(\frac{2}{x^2} + \frac{1}{x} - x^{3/2}\right) dx = \int_1^2 2x^{-2} + x^{-1} - x^{3/2}\,dx$

$$= 2\cdot\frac{1}{-1}x^{-1} + \ln|x| - \frac{2}{5}x^{5/2}\Big|_1^2$$

$$\boxed{\begin{array}{l}\text{Fact:} \\ \frac{d}{dx}\ln|x| = \frac{1}{x}\end{array}}$$

$$= -2x^{-1} + \ln|x| - \frac{2}{5}x^{5/2}\Big|_1^2$$



$$\left(-2\left(\tfrac{1}{2}\right) + \ln(2) - \tfrac{2}{5}(2)^{5/2}\right) - \left(-2\cdot 1 + \cancelto{0}{\ln(1)} - \tfrac{2}{5}(1)^{5/2}\right)$$

$$= -1 + \ln(2) - \tfrac{2}{5}(2)^{5/2} + 2 + \tfrac{2}{5}$$

$$= 1 + \ln(2) - \tfrac{2}{5}\left(2^{5/2} - 1\right) \;//$$

<u>eg</u>

$$\int_0^{\frac{\pi}{3}} 4\sec(u)\tan(u)\,du$$

$$= 4\sec(u)\Big|_0^{\frac{\pi}{3}} = 4\sec\left(\tfrac{\pi}{3}\right) - 4\sec(0)$$

$$= 4\cdot 2 - 4\cdot 1 = 4 \;//$$

$$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$$

$$\sec(0) = \frac{1}{\cos(0)} = 1$$

2. Substitution

Idea: Recognize when integrand (i.e. function you are integrating) comes from the chain rule.

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\int_a^b f'(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du = f(u)\Big|_{g(a)}^{g(b)}$$

$$= \underline{f(g(b)) - f(g(a))}$$

New variable:

$u = g(x)$ $\qquad x = a \to u = g(a)$

$\qquad\qquad\qquad x = b \to u = g(b)$

$\frac{du}{dx} = g'(x)$

$g'(x)\,dx = du$

Point: Look for a substitution $u = g(x)$

eg. $\displaystyle\int_0^{\pi} 3\cos^2(x)\sin(x)\,dx$

$\cancel{u = \cos^2(x)}$

$\cancel{du = -2\cos(x)\sin(x)dx}$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$x = 0 \to u = 1$

$x = \pi \to u = -1$

$$= -\int_0^{\pi} 3\cos^2(x)\left(-\sin(x)\,dx\right)$$

$$= -\int_1^{-1} 3u^2\,du$$

$$= \int_{-1}^{1} 3u^2\,du = u^3\Big|_{-1}^{1} = (1)^3 - (-1)^3$$

$$= 1 + 1 = 2 \,//$$

e.g. $\displaystyle \frac{1}{4}\int_{-1}^{0} \underbrace{\underbrace{4t^3}\underbrace{(1+t^4)^3}_{u^3}\underbrace{dt}}_{du}$

$u = 1 + t^4$

$du = 4t^3\,dt$

$t = -1 \quad u = 2$

$t = 0 \quad u = 1$

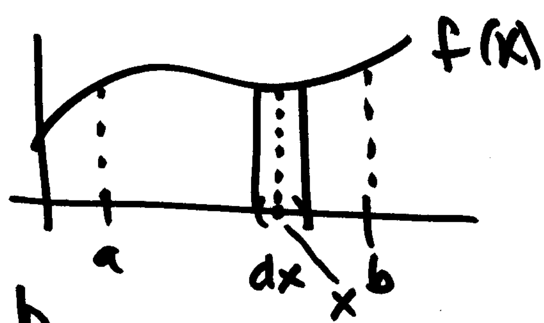
$$= \frac{1}{4}\int_{2}^{1} u^3\,du$$

$$= -\frac{1}{4}\int_{1}^{2} u^3\,du = -\frac{1}{4}\cdot\left(\frac{1}{4}u^4\right)\Big|_{1}^{2}$$

$$= -\frac{1}{16}\left(2^4 - 1^4\right) = -\frac{15}{16} \,//$$

This week we meet in <u>222</u>.

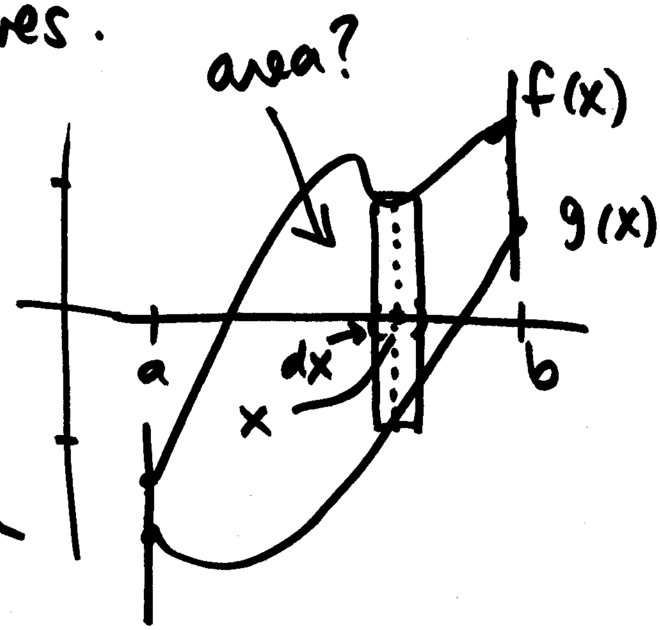
3. Areas between curves.



$$\int_a^b \underbrace{f(x)\,dx}_{\text{area element}} = \text{area between graph of } f(x) \text{ + } x\text{-axis.}$$

area element

sum of all area elements.



want area between graph of f(x) and graph of g(x).

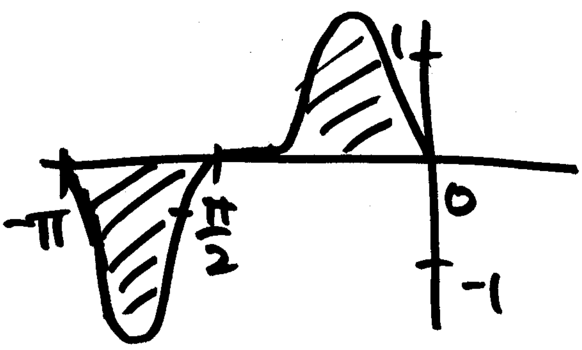
area element =

$$(f(x)-g(x))\,dx = dA$$

$$\text{area} = \int_a^b dA = \int_a^b (f(x)-g(x))\,dx$$

eg #50 p.411

$$y = \frac{\pi}{2}\cos(x)\left(\sin(\pi + \pi\sin(x))\right)$$



Want to find total area of region

$$\int_{-\pi}^{0} \frac{\pi}{2}\cos(x)\left(\sin(\pi + \pi\sin(x))\right)dx = 0$$

(NO GOOD)

$$A = -\int_{-\pi}^{-\frac{\pi}{2}} \frac{\pi}{2}\cos(x)\cdot\sin(\pi + \pi\sin(x))\,dx$$

$$+ \int_{-\frac{\pi}{2}}^{0} \frac{\pi}{2}\cos(x)\cdot\sin(\pi + \pi\sin(x))\,dx$$

$$= -\frac{1}{2}\int_{\pi}^{0}\sin(u)\,du + \frac{1}{2}\int_{0}^{\pi}\sin(u)\,du$$

$$= \frac{1}{2}\int_{0}^{\pi}\sin(u)\,du + \frac{1}{2}\int_{0}^{\pi}\sin(u)\,du = \int_{0}^{\pi}\sin(u)\,du$$

$$= -\cos(u)\Big|_{0}^{\pi} = -\cos(\pi) - (-\cos(0)) = -(-1) + 1 = 2$$

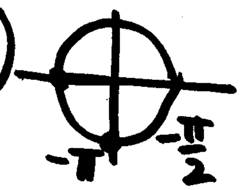
$$\frac{1}{2}\int_{-\pi}^{-\frac{\pi}{2}} 2\,\frac{\pi}{2}\cos(x)\cdot\sin(\pi+\pi\sin(x))\,dx$$

$$u=\pi+\pi\sin(x)$$

$$du=\pi\cos(x)\,dx$$

$$x=-\pi \qquad u=\pi+\pi(\overset{0}{\cancel{\sin(-\pi)}})=\pi$$

$$x=-\frac{\pi}{2} \qquad u=\pi+\pi\cdot\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1}$$



$$=0$$

$$=\frac{1}{2}\int_{\pi}^{0}\sin(u)\,du$$

$$\int_{-\frac{\pi}{2}}^{0}\frac{\pi}{2}\cos(x)\cdot\sin(\pi+\pi\sin(x))\,dx$$

same substitution
gives same integral
except.

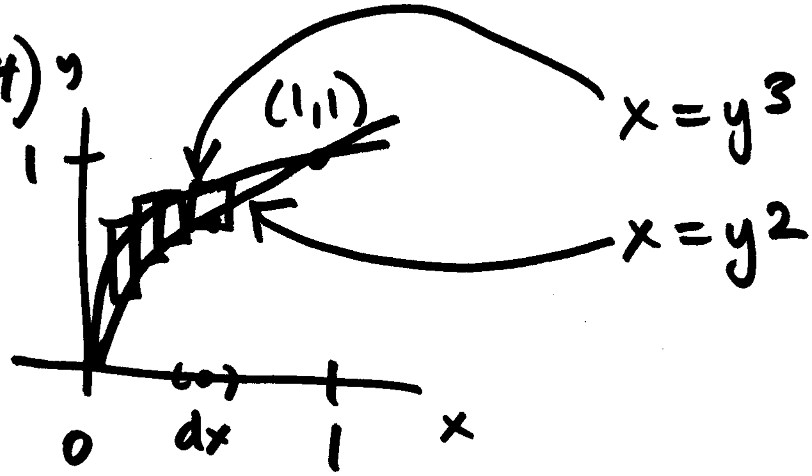
$$u=\pi+\pi\sin(x)$$

$$x=0 \qquad u=\pi+\pi\sin(0)=\pi$$

$$x=-\frac{\pi}{2} \qquad u=0$$

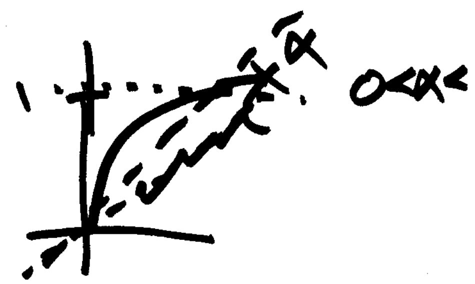
$$=\frac{1}{2}\int_{0}^{\pi}\sin(u)\,du$$

#54)



$y = x^{1/3}$

$y = x^{1/2}$



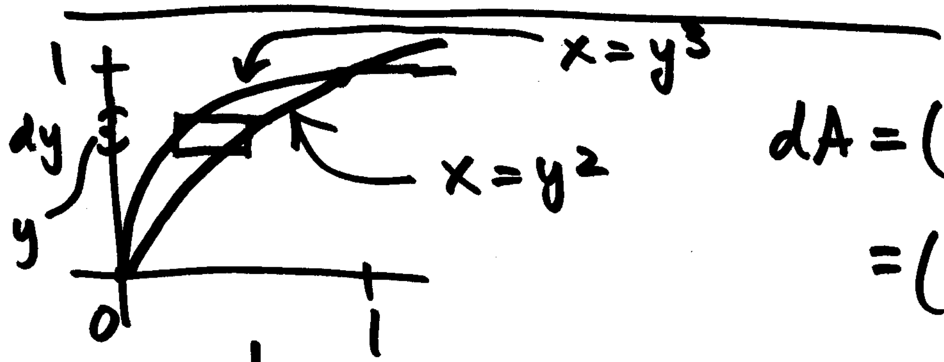
Area:

$$dA = (x^{1/3} - x^{1/2})\,dx$$

$$A = \int_0^1 dA = \int_0^1 (x^{1/3} - x^{1/2})\,dx$$

$$= \frac{3}{4}x^{4/3} - \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{3}{4}(1)^{4/3} - \frac{2}{3}(1)^{3/2} - 0$$

$$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$



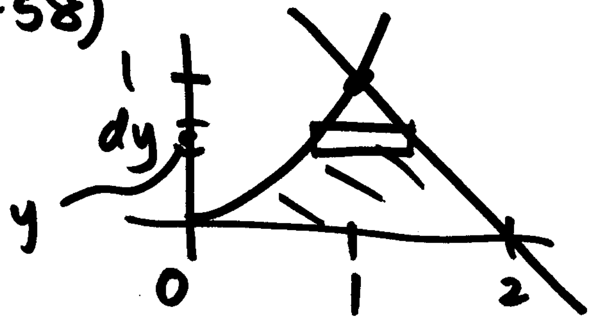
$$dA = (\text{width})\,dy$$

$$= (y^2 - y^3)\,dy$$

$$A = \int_0^1 dA = \int_0^1 (y^2 - y^3)\,dy = \frac{1}{3}y^3 - \frac{1}{4}y^4\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \;//$$

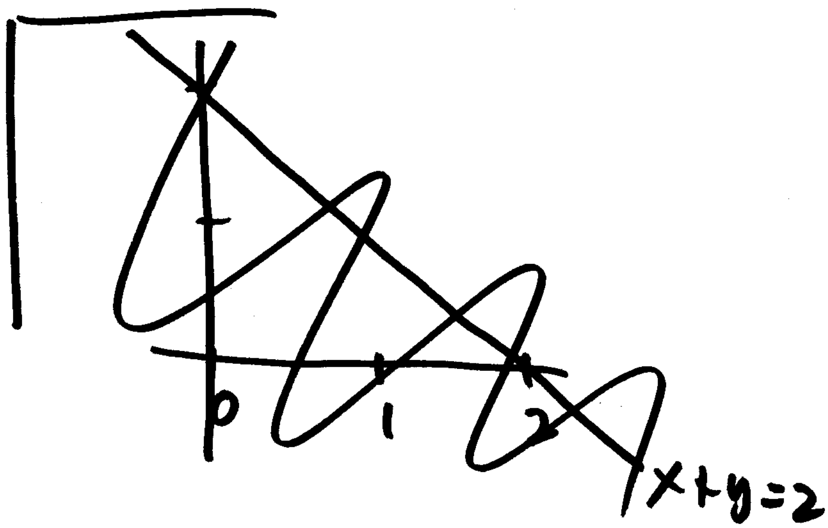
eg #58)

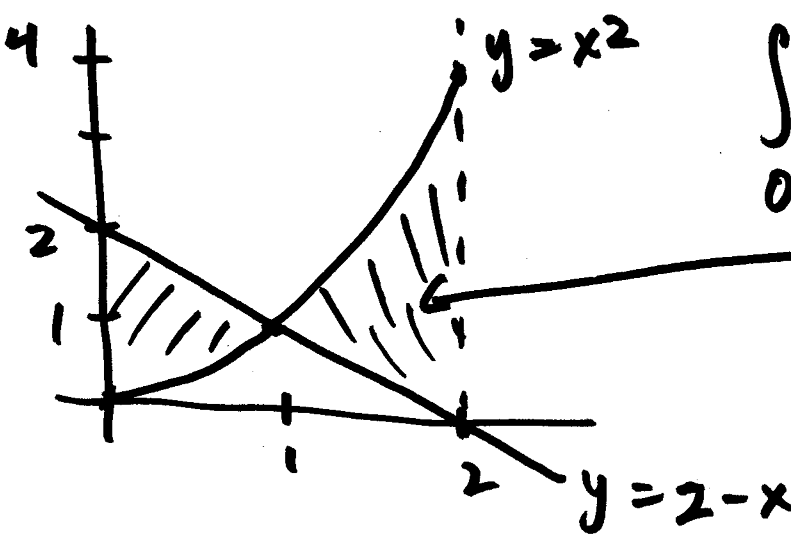


$y=x^2 \quad x=y^{1/2}$

$x+y=2 \quad x=2-y$

$\cancel{y=2-x}$





$$\int_0^2 x^2-(2-x)\,dx$$

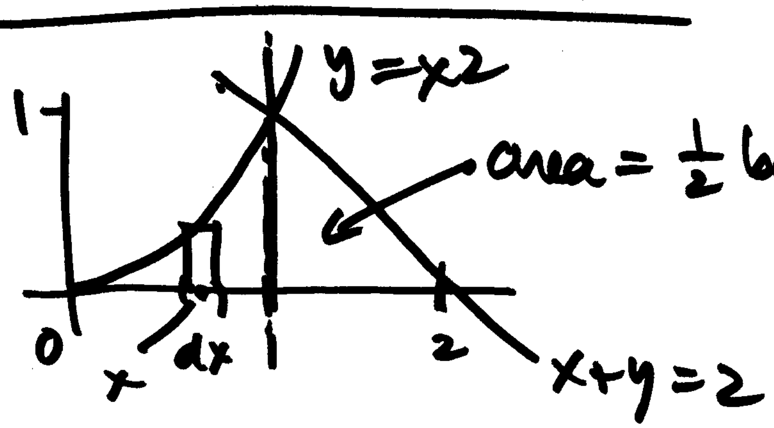
$$dA = (\text{width})\,dy$$

$$= (2 - y - y^{1/2})\,dy$$

$$A = \int_0^1 dA = \int_0^1 2 - y - y^{1/2}\,dy$$

$$= 2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\Big|_0^1$$

$$= 2 - \frac{1}{2} - \frac{2}{3} - 0 = \frac{3}{2} - \frac{2}{3} = \frac{5}{6} \quad //$$



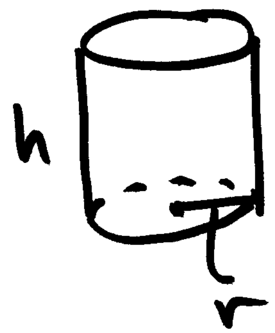
$$A = \int_0^1 x^2\,dx + \frac{1}{2} = \frac{1}{3}x^3\Big|_0^1 + \frac{1}{2}$$

$$= \frac{1}{3} - 0 + \frac{1}{2} = \frac{5}{6} \quad //$$

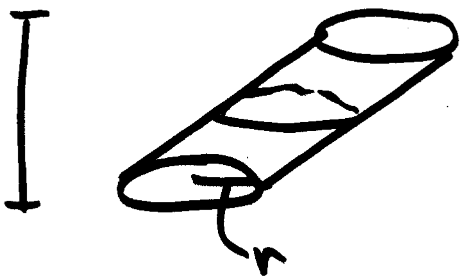
6.1 Volumes by slicing + rotation

1. Slicing

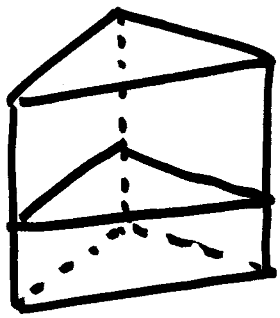
Idea: example:



$$V = \underbrace{\pi r^2}_{\text{area of base}} \underbrace{h}_{\text{height}}$$



$$V = \pi r^2 \cdot h$$

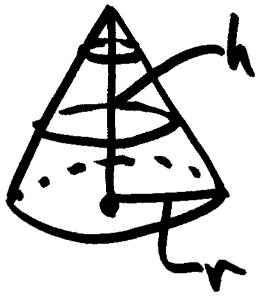


Vol = (area of each cross-section) × (height)

problem

Suppose that the area of a cross-section changes with its altitude.

eg



Need to use calculus for this.