MATH 114 – EXAM 4 – SOLUTIONS

1 (a) $a_1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

$a_2 = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}$

$a_3 = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}$

(b) $s_1 = a_1 = \frac{2}{15}$

$s_2 = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{3} - \frac{1}{7} = \frac{4}{21}$

$s_3 = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$

(c) $s_n = \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{2n+1}} - \frac{1}{2n+3}\right)$

$= \frac{1}{3} - \frac{1}{2n+3}$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{3} - \frac{1}{2n+3} = \frac{1}{3}.$

2. (a) $\sum_{n=0}^{\infty} (-1)^n \frac{5}{3^n}$ Converges by AST

and since it is a geometric series with $r = \frac{-1}{3}$.

$$= \sum_{n=0}^{\infty} 5 \cdot \left(\frac{-1}{3}\right)^n = \frac{5}{1-\left(-\frac{1}{3}\right)} = 5 \cdot \frac{3}{4} = \frac{15}{4} //$$

(b) $\sum_{n=1}^{\infty} 2^{1/n}$ Diverges because $\lim_{n\to\infty} 2^{1/n} = 1 \neq 0$.

(c) $\sum_{n=4}^{\infty} \frac{2}{\sqrt{n}}$ Diverges because it is a p-series with $p = \frac{1}{2} < 1$.

3. $0.212121\cdots = .21 + .0021 + .000021 + \cdots$

$$= \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \cdots$$

$$= 21\cdot\left(\frac{1}{100}\right) + 21\cdot\left(\frac{1}{100}\right)^2 + 21\left(\frac{1}{100}\right)^3 + \cdots$$

$$= \sum_{n=1}^{\infty} 21\left(\frac{1}{100}\right)^n = \sum_{n=1}^{\infty} \frac{21}{100}\left(\frac{1}{100}\right)^{n-1} = \frac{\frac{21}{100}}{1-\frac{1}{100}}$$

$$= \frac{21}{99} = \frac{7}{33} //$$

4. (a) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}}$ Compare with $\frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1/2}}}{\frac{n}{(n^3+1)^{1/2}}} = \lim_{n\to\infty} \frac{(n^3+1)^{1/2}}{n^{3/2}} = \lim_{n\to\infty} \left(\frac{n^3+1}{n^3}\right)^{1/2} = 1$$

$\therefore$ series <u>diverges</u> with $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ by limit comp. test.

(b) $\sum_{n=2}^{\infty} \frac{1}{1+3^n}$ Compare with $\frac{1}{3^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{3^n}}{\frac{1}{1+3^n}} = \lim_{n\to\infty} \frac{1+3^n}{3^n} = \lim_{n\to\infty} \frac{1}{3^n}+1 = 1$$

$\therefore$ series converges with $\sum_{n=2}^{\infty} \frac{1}{3^n}$ by Limit comparison test.

5. (a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

Ratio test: $\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \frac{10^{n+1}}{10^n} \cdot \frac{n!}{(n+1)!}$

$= \frac{10}{n+1} \to 0$ as $n \to \infty$

Since $0<1$ series converges.

(b) $\sum_{n=3}^{\infty} \frac{n^3 2^n}{3^n}$

Ratio test:

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3 2^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^3 2^n} = \left(\frac{n+1}{n}\right)^3 \cdot \frac{2}{3} \to \frac{2}{3}$ as $n \to \infty$

$\therefore$ Since $\frac{2}{3}<1$ series converges.

Root test:

$(a_n)^{1/n} = \left(\frac{n^3 2^n}{3^n}\right)^{1/n} = \frac{n^{3/n} \cdot 2}{3} = \frac{(n^{1/n})^3 \cdot 2}{3} \to \frac{2}{3}$ as $n \to \infty$

Since $\frac{2}{3}<1$ series converges.

6. (a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n^{2/5}}$ converges by AST

since $\frac{3}{n^{2/5}}$ decreases to zero as $n \to \infty$.

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{3}{n^{2/5}} \right| = \sum_{n=1}^{\infty} \frac{3}{n^{2/5}}$ diverges since it is

a p-series with $p = \frac{2}{5} < 1$.

$\therefore$ series <u>converges</u> <u>conditionally</u>.

(b) $\sum_{n=0}^{\infty} \frac{(-3)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{5}\right)^n$

$\sum_{n=0}^{\infty} \left| (-1)^n \left(\frac{3}{5}\right)^n \right| = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$ converges since it is

a geometric series with $r = \frac{3}{5} < 1$

$\therefore$ series <u>converges</u> <u>absolutely</u>.