MATH 114 - EXAM 3 - SOLUTIONS

1. (a) $\int_0^1 (1-x^2)^{1/2}\,dx$

$$x = \sin\theta$$
$$dx = \cos\theta\, d\theta$$
$$1-x^2 = 1-\sin^2\theta = \cos^2\theta$$
$$(1-x^2)^{1/2} = (\cos^2\theta)^{1/2} = \cos\theta$$
$$x = 0 \quad \theta = 0$$
$$x = 1 \quad \theta = \frac{\pi}{2}$$

$$= \int_0^{\pi/2} \cos\theta \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\, d\theta \quad //$$

(b) $\int \frac{x^2}{(x^2-1)^{3/2}}\,dx$

$$x = \sec\theta$$
$$dx = \sec\theta\tan\theta\, d\theta$$
$$x^2 = \sec^2\theta$$
$$x^2-1 = \sec^2\theta - 1 = \tan^2\theta$$
$$(x^2-1)^{3/2} = (\tan^2\theta)^{3/2} = \tan^3\theta$$

$$= \int \frac{\sec^2\theta}{\tan^3\theta}\cdot \sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\sec^3\theta}{\tan^2\theta}\, d\theta \quad //$$

$$= \int \frac{1}{\cos^3\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \frac{d\theta}{\cos\theta\sin^2\theta} \quad //$$

2. (a) $\int \frac{x}{(2x+1)^{3/2}} = \int x(2x+1)^{-3/2}\,dx$

#1 with $a=2$, $b=1$, $n=-3/2$

$$= \frac{(2x+1)^{-1/2}}{4}\left[\frac{2x+1}{1/2} - \frac{1}{-1/2}\right] + c$$

$$= \frac{1}{2}(2x+1)^{1/2} + \frac{1}{2}(2x-1)^{-1/2} + c \,//.$$

(b) $\int \frac{dx}{(4+x^2)^2}$ #3 with $a=2$

$$= \frac{x}{8(4+x^2)} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + c \,//$$

3. (a) $\int_2^\infty x e^{-x}\,dx = \lim_{b\to\infty} \int_2^b x e^{-x}\,dx$

$\int_2^b x e^{-x}\,dx$ $\quad u = x \quad dv = e^{-x}dx$

$\quad du = dx \quad v = -e^{-x}$

$$= -xe^{-x}\Big|_2^b + \int_2^b e^{-x}\,dx$$

$$= -be^{-b} + 2e^{-2} - e^{-x}\Big|_2^b$$

$$= -be^{-b} + 2e^{-2} - e^{-b} + e^{-2}$$

$$= \lim_{b\to\infty} -\cancel{be^{-b}}^{\,0} + 2e^{-2} - \cancel{e^{-b}}^{\,0} + e^{-2} = 2e^{-2} + e^{-2}$$

$$= 3e^{-2}$$ //.

(b) $$\int_0^1 \frac{dx}{(1-x)^{1/2}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(1-x)^{1/2}}$$

$$= \lim_{b\to 1^-} \left(-2(1-x)^{1/2} \Big|_0^b \right) = \lim_{b\to 1^-} -2\cancel{(1-b)}^{\,0\;1/2} + 2 = 2$$ //

4. (a) $$\int_1^\infty \frac{x+1}{x(x^2+1)}\,dx$$ compare with $$\int_1^\infty \frac{1}{x^2}\,dx$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{x+1}{x(x^2+1)}} = \lim_{x\to\infty} \frac{x(x^2+1)}{x^{\cancel{2}}(x+1)} = \lim_{x\to\infty} \frac{x^2+1}{x^2+x} = 1$$

$\therefore \int_1^\infty \frac{x+1}{x(x^2+1)}\,dx$ converges with $\int_1^\infty \frac{1}{x^2}\,dx$ //

(b) $$\int_1^\infty \frac{y+4}{y(y+1)}\,dy$$ Compare with $$\int_1^\infty \frac{1}{y}\,dy$$

$$\lim_{y\to\infty} \frac{\frac{1}{y}}{\frac{y+4}{y(y+1)}} = \lim_{y\to\infty} \frac{\cancel{y}(y+1)}{\cancel{y}(y+4)} = 1$$

$\therefore \int_1^\infty \frac{y+4}{y(y+1)}\,dy$ diverges with $\int_1^\infty \frac{1}{y}\,dy$ //

5. (a) $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta$

OR $A = 2\int_0^{\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta$

$= \int_0^{\pi} (1+\cos\theta)^2 \, d\theta$ //

(b) $L = \int_0^{2\pi} \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{1/2} d\theta \qquad \frac{dr}{d\theta} = -\sin\theta$

$= \int_0^{2\pi} \left((1+\cos\theta)^2 + (-\sin\theta)^2\right)^{1/2} d\theta$

$= \int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\right)^{1/2} d\theta$

$= \int_0^{2\pi} (2 + 2\cos\theta)^{1/2} \, d\theta.$ //