Exam 2

3.1 – 3.6 (omit 1.5, 1.6)

3.2 25) 3.4 19) 3.5 37) 83)

25) $v = \dfrac{1+x-4\sqrt{x}}{x} = (1+x-4x^{1/2})x^{-1}$

Quotient: $\dfrac{dv}{dx} = \dfrac{x\frac{d}{dx}(1+x-4x^{1/2}) - (1+x-4x^{1/2})\frac{d}{dx}(x)}{x^2}$

$= \dfrac{x(1-4\cdot\frac{1}{2}x^{-1/2}) - (1+x-4x^{1/2})}{x^2}$

$= \dfrac{\cancel{x} - 2x^{1/2} - 1 \cancel{-x} + 4x^{1/2}}{x^2} = \dfrac{2x^{1/2}-1}{x^2}$

Product: $\dfrac{dv}{dx} = (1+x-4x^{1/2})\dfrac{d}{dx}(x^{-1})$
$\quad + (x^{-1})\dfrac{d}{dx}(1+x-4x^{1/2})$

$= (1+x-4x^{1/2})(-x^{-2}) + (x^{-1})(1-2x^{-1/2})$

$= -x^{-2} \cancel{-x^{-1}} + 4x^{-3/2} \cancel{+x^{-1}} - 2x^{-3/2}$

$= (2x^{-3/2} - x^{-2})\dfrac{x^2}{x^2} = \dfrac{2x^{1/2}-1}{x^2}$

Simplify: $v = \dfrac{1+x-4x^{1/2}}{x} = \dfrac{1}{x} + 1 - 4x^{-1/2}$

$$= x^{-1} + 1 - 4x^{-1/2}$$

$$\frac{dv}{dx} = -x^{-2} - 4\left(-\tfrac{1}{2}x^{-3/2}\right)$$

$$= -x^{-2} + 2x^{-3/2} = \frac{2x^{1/2}-1}{x^2}$$

3.4 19) $r = \sec(\theta)\csc(\theta) = \dfrac{1}{\sin(\theta)\cos(\theta)}$

$$\frac{dr}{d\theta} = \sec(\theta)\frac{d}{d\theta}\csc(\theta) + \csc(\theta)\frac{d}{d\theta}\sec(\theta)$$

$$= \sec(\theta)\left(-\csc(\theta)\cot(\theta)\right) + \csc(\theta)\left(\sec(\theta)\tan(\theta)\right)$$

$$= -\sec(\theta)\csc(\theta)\cot(\theta) + \csc(\theta)\sec(\theta)\tan(\theta)$$

$$= \sec(\theta)\csc(\theta)\left(\tan(\theta) - \cot(\theta)\right).$$

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$$\sec(x)\left(\sec^2(x) + \tan^2(x)\right) = \sec(x)\left(2\sec^2(x) - 1\right)$$

$$\cos^2(x) + \sin^2(x) = 1$$

$$1 + \tan^2(x) = \sec^2(x) \checkmark$$

$$\cot^2(x) + 1 = \csc^2(x)$$

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3.5 37) $y = (x^2-2x+2)e^{5x/2}$

$$y' = (x^2-2x+2)\frac{d}{dx}(e^{5x/2}) + e^{5x/2}\frac{d}{dx}(x^2-2x+2)$$

$$= (x^2-2x+2)\left(e^{5x/2}\cdot\frac{5}{2}\right) + e^{5x/2}(2x-2)$$

$$= e^{5x/2}\left(\frac{5}{2}x^2 - 5x + 5 + 2x - 2\right)$$

$$= e^{5x/2}\left(\frac{5}{2}x^2 - 3x + 3\right) \;//$$

$$e^{5x/2} = e^{\frac{5}{2}\cdot x} \xrightarrow{\frac{d}{dx}} e^{\frac{5}{2}x}\cdot\frac{5}{2}$$

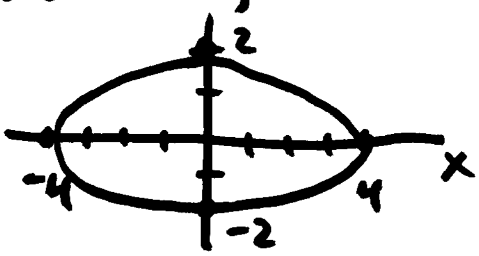
$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

83) $x = 4\cos t \quad y = 2\sin t \quad 0 \le t \le 2\pi$

use: $\cos^2 t + \sin^2 t = 1$ $\qquad \frac{x^2}{16} + \frac{y^2}{4} = 1$

$\frac{x}{4} = \cos t \quad \frac{y}{2} = \sin t$ $\qquad$ ellipse

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$



Derivatives of inverse functions (cont'd)

Example: $\ln(x)$ - inverse of $e^x$

$\ln(x) = y \iff e^y = x$

$e^{\ln(x)} = x \qquad \ln(e^x) = x$

$$\frac{d}{dx}\left(e^{\ln(x)}\right) = \frac{d}{dx}(x)$$

$$e^{\ln(x)} \cdot \frac{d}{dx}\ln(x) = 1$$

$$x \cdot \frac{d}{dx}\ln(x) = 1$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Chain rule form:

$$\boxed{\frac{d}{dx}\ln(u) = \frac{1}{u}\cdot\frac{du}{dx}}$$

Laws of logarithms:

$\ln(xy) = \ln(x) + \ln(y)$

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$\ln(x^y) = y\ln(x)$

e.g. $y = \ln(t^{3/2})$ Find $y'$

Chain rule : $y' = \frac{1}{t^{3/2}} \frac{d}{dt}(t^{3/2})$

$\boxed{t^{1/2} \cdot t^{-3/2} = t^{-1}}$

$= \frac{1}{t^{3/2}} \cdot \frac{3}{2} t^{1/2} = \frac{3}{2} \frac{t^{1/2}}{t^{3/2}} = \frac{3}{2} \frac{1}{t}$

Simplify: $y = \ln(t^{3/2}) = \frac{3}{2} \ln(t)$

$y' = \frac{3}{2} \frac{1}{t}$

e.g. $y = \ln\left(\frac{10}{x}\right)$

Simplify : $y = \ln\left(\frac{10}{x}\right) = \ln(10) - \ln(x)$

$y' = -\frac{1}{x}$

Chain: $y' = \frac{x}{10} \cdot \frac{d}{dx}\left(\frac{10}{x}\right) = \frac{x}{\cancel{10}}(-\cancel{10}x^{-2})$

$= -x \cdot x^{-2} = -x^{-1} = -\frac{1}{x}.$

$\underline{\text{e.g.}}$ $y = (\ln(x))^3$

$\ln(x^3) = 3\ln(x)$ $\quad \underline{(\ln(x))^3}$

$y' = 3(\ln(x))^2 \cdot \frac{1}{x}$

$= \frac{3}{x}(\ln(x))^2.$

$\underline{\text{eg}}.$ $y = \frac{1+\ln(t)}{t}$

Simplify: $y = \frac{1}{t} + \frac{\ln(t)}{t} = t^{-1} + t^{-1}\ln(t)$

$y' = -t^{-2} + t^{-1}\frac{d}{dt}\ln(t) + \ln(t)\frac{d}{dt}(t^{-1})$

$= -t^{-2} + t^{-1}\cdot\frac{1}{t} + \ln(t)(-t^{-2})$

$= \cancel{-t^{-2}} + \cancel{t^{-2}} - t^{-2}\ln(t)$

$= -t^{-2}\ln(t) = \frac{-\ln(t)}{t^2}.$

Quotient: $y' = \frac{t\frac{d}{dt}(1+\ln t) - (1+\ln t)\frac{d}{dt}(t)}{t^2}$

$$= \frac{t \cdot \frac{1}{t} - (1+\ln t)(1)}{t^2}$$

$$= \frac{\cancel{1} - \cancel{1} - \ln(t)}{t^2} = \frac{-\ln(t)}{t^2}.$$

e.g #38

$$\boxed{\ln(x+y) \neq \ln(x) + \ln(y)}$$

$$y = \ln\left(\frac{\sqrt{\sin\theta \cos\theta}}{1+2\ln(\theta)}\right)$$

$$y' = \frac{1+2\ln(\theta)}{\sqrt{\sin\theta\cos\theta}} \cdot \frac{d}{d\theta}\left(\frac{\sqrt{\sin\theta\cos\theta}}{1+2\ln(\theta)}\right) \text{ forget it.}$$

Simplify:

$$y = \ln(\sqrt{\sin\theta\cos\theta}) - \ln(1+2\ln\theta)$$

$$= \frac{1}{2}\ln(\sin\theta\cos\theta) - \ln(1+2\ln\theta)$$

$$= \frac{1}{2}\ln(\sin\theta) + \frac{1}{2}\ln(\cos\theta) - \ln(1+2\ln\theta)$$

$$y' = \frac{1}{2}\frac{1}{\sin\theta}\cdot\cos\theta + \frac{1}{2}\frac{1}{\cos\theta}\cdot -\sin\theta - \frac{2\cdot\frac{1}{\theta}}{1+2\ln\theta}$$

$$y' = \frac{1}{2}\cot\theta - \frac{1}{2}\tan\theta - \frac{2}{\theta(1+2\ln\theta)}$$

Logarithmic Differentiation

$$y = \sqrt{(x^2+1)(x-1)^2}$$

$$\ln(y) = \ln\left(\sqrt{(x^2+1)(x-1)^2}\right)$$

$$= \frac{1}{2}\ln\left((x^2+1)(x-1)^2\right)$$

$$= \frac{1}{2}\left(\ln(x^2+1) + \ln[(x-1)^2]\right)$$

$$= \frac{1}{2}\ln(x^2+1) + \ln(x-1)$$

use implicit diff.

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left[\frac{1}{2}\ln(x^2+1) + \ln(x-1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+1} + \frac{1}{x-1} = \frac{x}{x^2+1} + \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left(\frac{x}{x^2+1} + \frac{1}{x-1}\right) = \sqrt{(x^2+1)(x-1)^2}\left(\frac{x}{x^2+1} + \frac{1}{x-1}\right)$$

eg $$y = \frac{\theta \sin\theta}{\sqrt{\sec\theta}}$$

$$\ln(y) = \ln\left(\frac{\theta \sin\theta}{\sqrt{\sec\theta}}\right)$$

$$= \ln(\theta \sin\theta) - \ln(\sqrt{\sec\theta})$$

$$= \ln\theta + \ln(\sin\theta) - \frac{1}{2}\ln(\sec\theta)$$

$$\frac{1}{y}\frac{dy}{d\theta} = \frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{1}{2}\,\frac{\cancel{\sec}\theta \tan\theta}{\cancel{\sec}(\theta)}$$

$$\frac{dy}{d\theta} = y\left(\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right)$$

$$= \frac{\theta \sin\theta}{\sqrt{\sec\theta}}\left(\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right)$$

Derivative of $a^x$ , $a>0$

Recall:

$$\lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = a^x \underbrace{\lim_{h\to 0} \frac{a^h-1}{h}}_{?}$$

$\ln(a)$

Simplify: $$a^x = e^{\ln(a^x)} = e^{x\ln(a)}$$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln(a)}\right) = e^{x\ln(a)} \cdot \frac{d}{dx}(x\ln(a))$$

$$= a^x \frac{d}{dx}(x\ln(a)) = \ln(a)\cdot a^x$$

$$\frac{d}{dx}(a^x) = \ln(a)\cdot a^x$$

Chain rule version

$$\boxed{\frac{d}{dx} a^u = \ln(a)\cdot a^u \cdot \frac{du}{dx}}$$

Derivative of $\log_a(x)$. $\quad y = \log_a(x) \iff a^y = x$

<u>Idea</u>: Know

$$a^{\log_a(x)} = x$$

$$\frac{d}{dx}\left(a^{\log_a(x)}\right) = \frac{d}{dx}(x)$$

$$\ln(a)\, \underbrace{a^{\log_a(x)}}_{x} \cdot \frac{d}{dx} \log_a(x) = 1$$

$$x \ln(a) \cdot \frac{d}{dx} \log_a(x) = 1$$

$$\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$$

$$\boxed{\frac{d}{dx} \log_a(u) = \frac{1}{u \ln(a)} \cdot \frac{du}{dx}}$$

<u>eg</u> $\quad y = 3^{-t^2}$

$$y' = \ln(3) \cdot 3^{-t^2} \frac{d}{dt}(-t^2)$$

$$= \ln(3) \cdot 3^{-t^2}(-2t) = -2t \ln(3)\, 3^{-t^2}$$

e.g. $y = \log_5(e^x) - \log_5(\sqrt{x})$

$$= x \log_5(e) - \frac{1}{2}\log_5(x)$$

$$y' = \log_5(e) - \frac{1}{2}\frac{1}{x\ln(5)}$$

e.g. $y = x^x$

Logarithmic diff:

$$\ln(y) = \ln(x^x) = x\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln(x)\,(1)$$

$$\frac{dy}{dx} = y\left(1 + \ln(x)\right) = x^x\left(1 + \ln(x)\right).$$

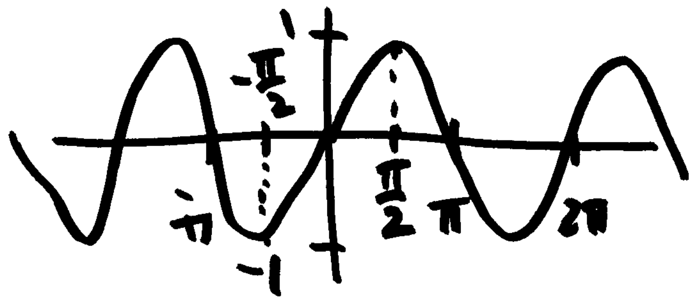
Another way:

$$y = x^x = e^{\ln(x^x)} = e^{x\ln(x)}$$

$$y' = e^{x\ln(x)}\frac{d}{dx}\left(x\ln(x)\right) = x^x\left(1 + \ln(x)\right).$$

# 3.8 Inverse Trig Functions

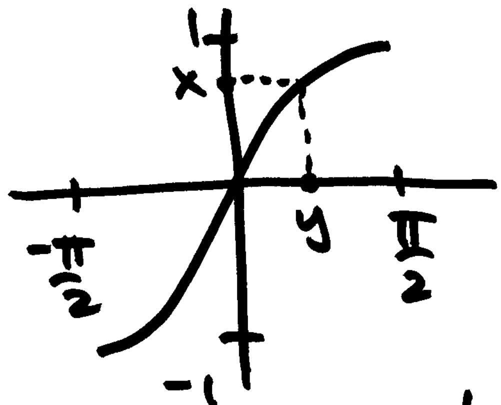
e.g. $\sin^{-1}(x) = \arcsin(x)$




not one-to-one.

Idea: Restrict the domain so that it is one-to-one

$\sin(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$




one-to-one!

$y = \sin^{-1}(x) \iff \sin(y) = x$

$-1 \le x \le 1$ $\qquad$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Think of: $x$ is a "number" between $-1$ and $1$

$y = \sin^{-1}(x)$ is an "angle" between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

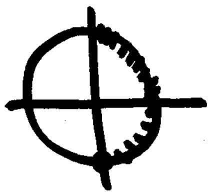
e.g. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = y \iff \sin(y) = \frac{\sqrt{2}}{2}$

$y = \frac{\pi}{4}$

$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$

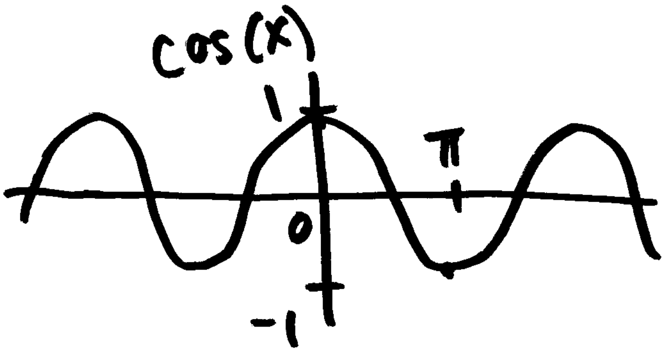
$\sin^{-1}(-1) = y \iff \sin(y) = -1$

$y = -\frac{\pi}{2}$



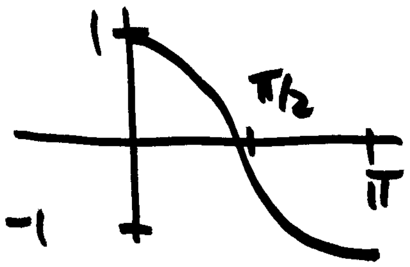
$\sin^{-1}(-1) = -\frac{\pi}{2}.$

e.g. $\cos^{-1}(x) = \arccos(x)$



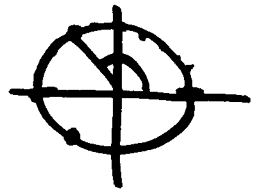

Restrict domain

$\cos(x),\ 0 \le x \le \pi$




$y = \cos^{-1}(x) \iff \cos(y) = x$

$-1 \le x \le 1 \qquad 0 \le y \le \pi$



$\cos^{-1}\left(-\frac{1}{2}\right) = y \qquad \cos(y) = -\frac{1}{2}$

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \qquad y = \frac{2\pi}{3}$