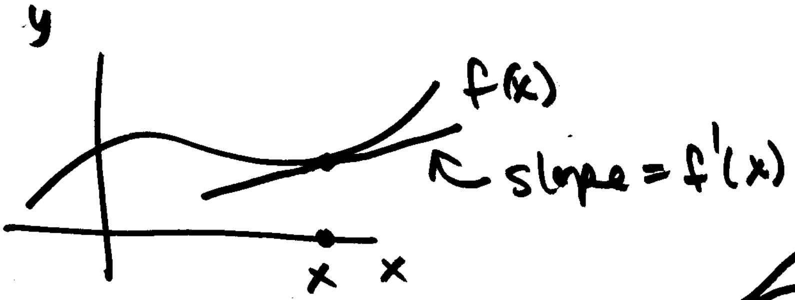
MAPLE #1 due tomorrow.

Chain rule $\qquad \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

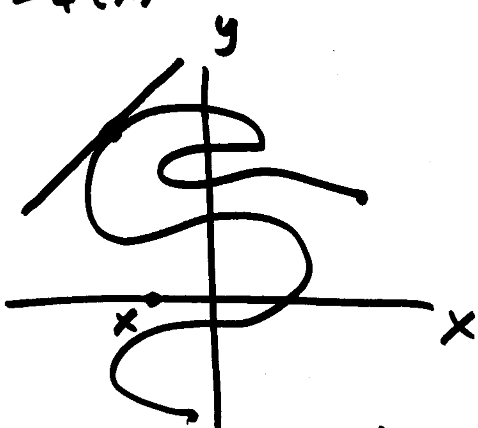
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Parametric equations

Idea: Given a curve $y = f(x)$, we can find tangent lines at each point of curve



What if curve looks like:



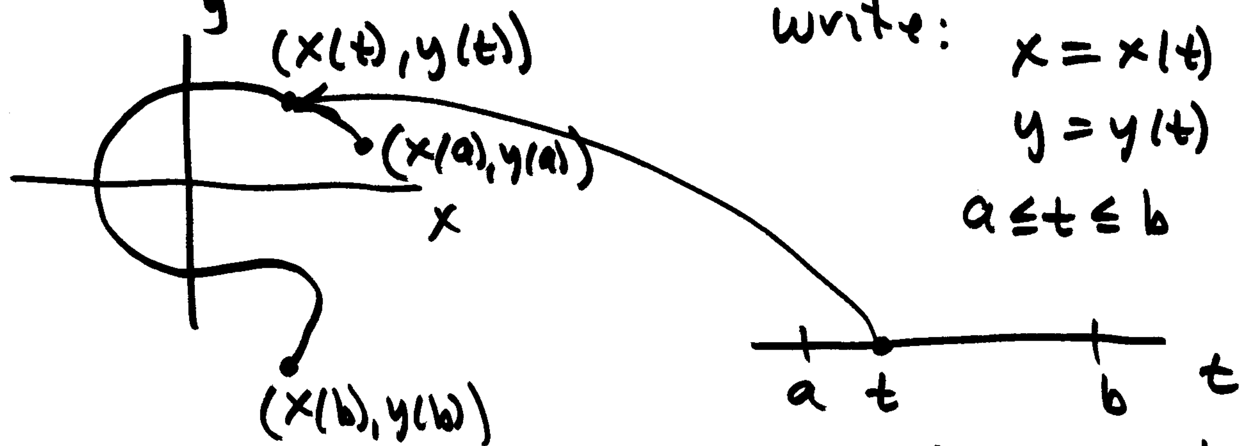
This cannot be written as $y = f(x)$.

But curve still has tangents at each point.

Need: (1) A new way to describe curves.

(2) A way to find tangents of such curves.

parametric equations:



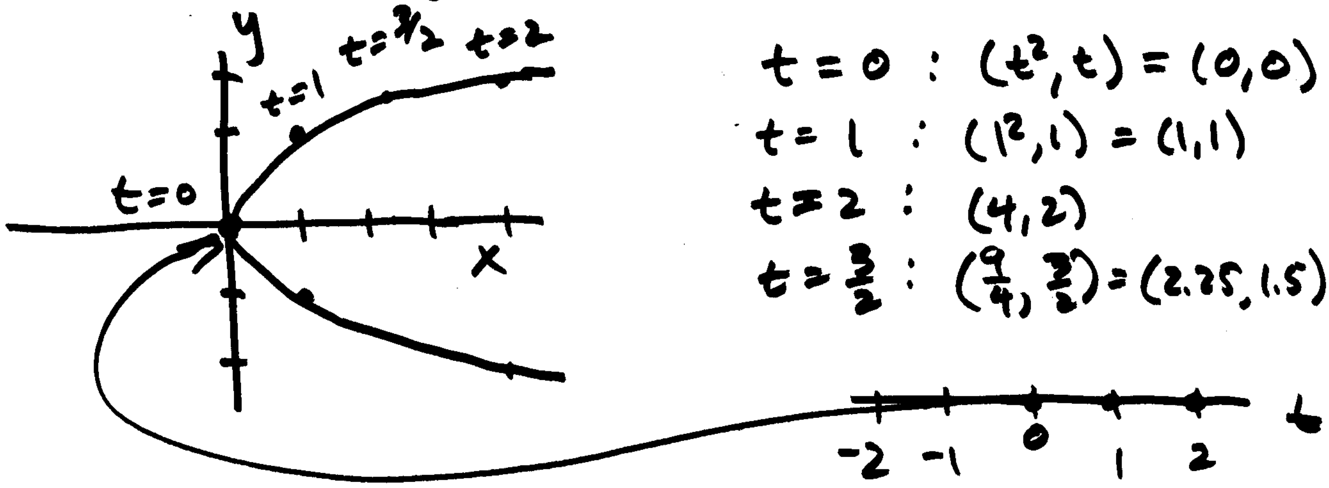
write: $x = x(t)$

$y = y(t)$

$a \le t \le b$

axis of parameters.

<u>e.g.</u> $x = t^2 \quad y = t \qquad -2 \le t \le 2$



$t = 0 : (t^2, t) = (0,0)$

$t = 1 : (1^2, 1) = (1,1)$

$t = 2 : (4,2)$

$t = \frac{3}{2} : \left(\frac{9}{4}, \frac{3}{2}\right) = (2.25, 1.5)$

$t = -1 : (1,-1)$

$t = -2 : (4,-2)$

How else to describe curve?
Eliminate parameter.

$x = t^2 \quad y = t$

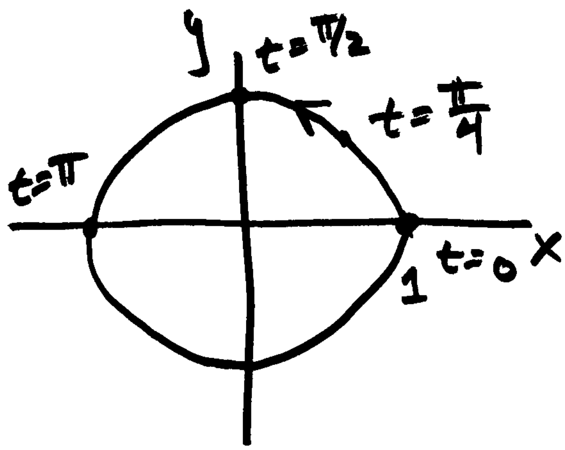
$\downarrow$

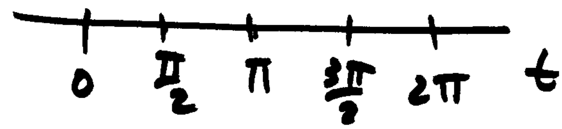
$y^2 = t^2$

$x = t^2 = y^2 \quad \therefore \quad x = y^2$

parabola opening to right

e.g. $x = \cos(t) \quad y = \sin(t) \qquad 0 \le t \le 2\pi$





$t=0: \ (\cos(0), \sin(0)) = (1,0)$

$t=\frac{\pi}{2}: \ (\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2})) = (0,1)$

$t=\frac{\pi}{4}: \ (\cos\frac{\pi}{4}, \sin\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \approx (.7,.7)$

etc...

Eliminate parameter: $\cos^2(t) + \sin^2(t) = 1$

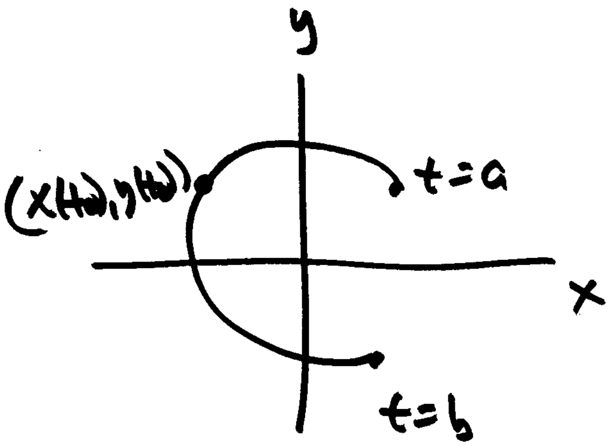
$x^2 + y^2 = 1$ circle

$x = \pm\sqrt{1-y^2}$

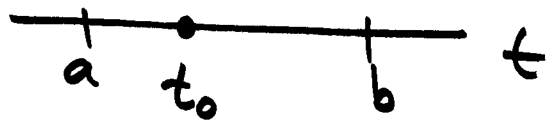
e.g. Any function $y = f(x)$ has a parametric representation:

$x = t \quad y = f(t)$.

What about tangents?

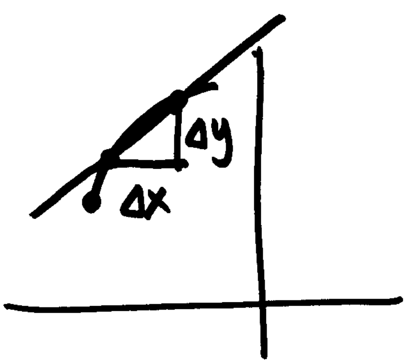


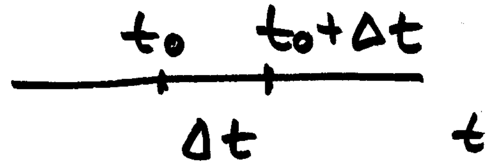
$x = x(t)$
$y = y(t)$



Recall: Derivative measures rate of change of "y" with respect to "x".

Want $\dfrac{\Delta y}{\Delta x}$ $\leftarrow$ change in y, $\leftarrow$ change in x





Want to relate Δx and Δy to Δt.

Look at Δx: $x = x(t)$

$$\frac{\Delta x}{\Delta t} = \frac{x(t_0+\Delta t) - x(t_0)}{\Delta t} \longrightarrow \frac{dx}{dt}$$

as $\Delta t \to 0$

This means $\dfrac{\Delta x}{\Delta t} \approx \dfrac{dx}{dt}$ when Δt small.

$\therefore \ \Delta x \approx \frac{dx}{dt} \Delta t$, Δt small

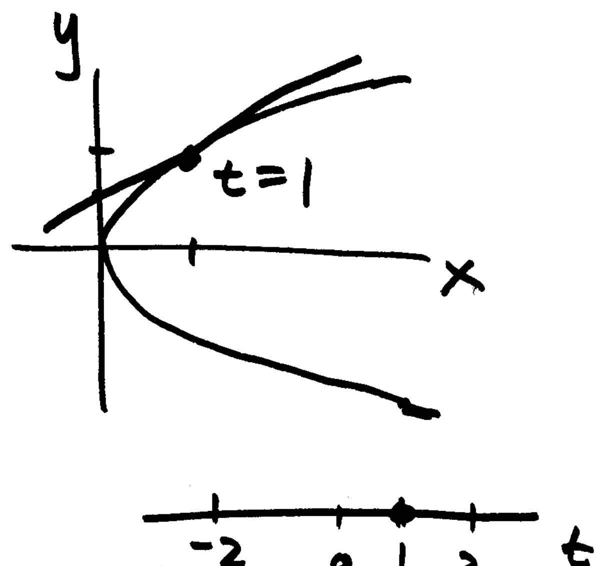
In the same way

$$\Delta y \approx \frac{dy}{dt} \Delta t \qquad \Delta t \text{ small.}$$

$\frac{\Delta y}{\Delta x} \approx \frac{dy/dt}{dx/dt}$. As $\Delta t \to 0$, the $\approx$ becomes $=$.

$$\therefore \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{slope of tangent line to curve } \begin{matrix} x = x(t) \\ y = y(t) \end{matrix}}$$

<u>e.g.</u> $x = t^2 \quad y = t$



$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 1$

$\therefore \ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2t}$

Find eqn of tangent line at $t=1$.

$t = 1 : (t^2, t) = (1,1)$ //

slope : $\frac{1}{2(1)} = \frac{1}{2}$ //

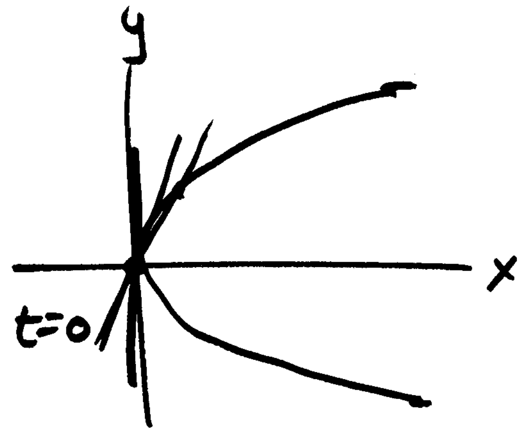
$y - 1 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}x + \frac{1}{2}$

How about $t=0$?

$$\lim_{t \to 0^\pm} \frac{1}{2t} = \pm\infty$$

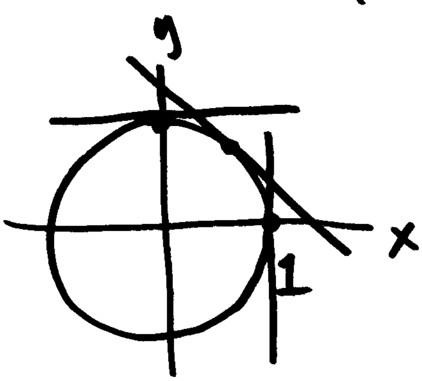
interpreted as a vertical tangent at $t=0$.



eg $x = \cos(t)$ $y = \sin(t)$ $0 \le t \le 2\pi$

Find $\frac{dy}{dx}$. $\frac{dy}{dt} = \cos(t)$ $\frac{dx}{dt} = -\sin(t)$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t)}{-\sin(t)} = -\cot(t)$$



$t=0$: vertical tangent

$t = \frac{\pi}{4}$: $\frac{dy}{dx} = -\cot\left(\frac{\pi}{4}\right)$

$= -1$

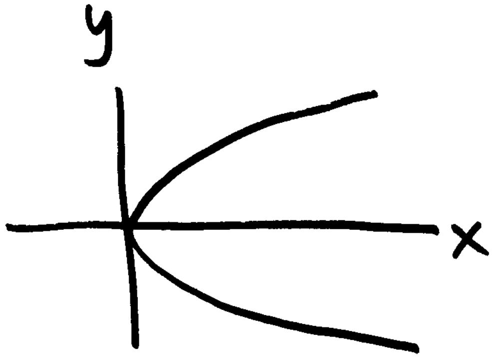
$t = \frac{\pi}{2}$: $\frac{dy}{dx} = -\cot\left(\frac{\pi}{2}\right) = 0$

3.6 Implicit Differentiation

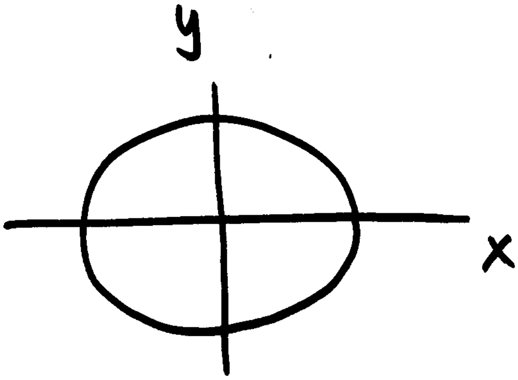
Idea: We can define curves by: $y = f(x)$

or $x = x(t)$, $y = y(t)$ or equations involving x and y.

e.g. $x = y^2$

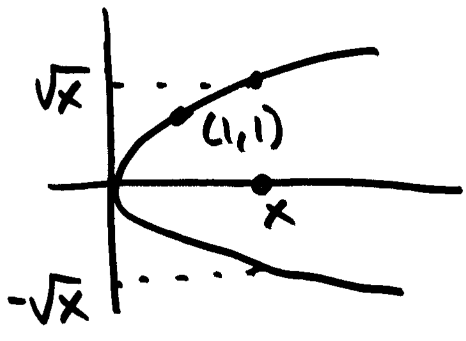


$x^2 + y^2 = 1$



Want to find tangents to such curves.

Idea: e.g. $x = y^2$



Not graph of function.

Why not? $x = y^2$

$y = \pm\sqrt{x}$

Can say curve is given by 2 functions

$y = \sqrt{x}$ when $y > 0$ $\quad y = -\sqrt{x}$ when $y < 0$.

Spse I want slope of tangent at $(1,1)$ //

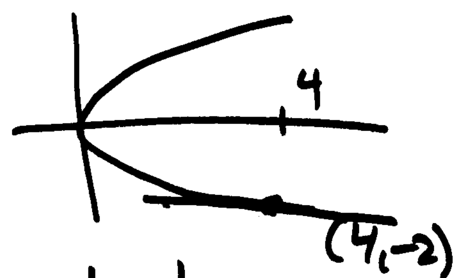
Around this point, curve is given by $\underline{y=\sqrt{x}}$

$\therefore y' = \frac{1}{2\sqrt{x}}$ $\therefore$ slope is $y'(1) = \frac{1}{2}$ //

What about slope at $(4,-2)$?

Around this point, curve is $y = -\sqrt{x}$ so

$y' = \frac{-1}{2\sqrt{x}}$ so $y'(4) = -\frac{1}{4}$



Easier way: $\boxed{\frac{d}{dx}(u^2) = 2u\frac{du}{dx}}$

$x = y^2$

Think of y as <u>some</u> function of x.

Then take derivative w.r.t. x of both sides

$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$

$$1 = 2y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

Slope of tangent at $(1,1)$

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{2}$ //

slope of tangent at $(4,-2)$

$\left.\frac{dy}{dx}\right|_{(4,-2)} = \frac{1}{2(-2)} = -\frac{1}{4}$ //

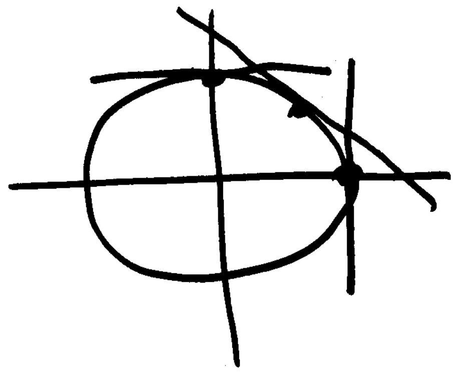
e.g. $x^2+y^2=1$. Find slope of tangent at $(0,1)$, $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $(1,0)$

Think of y as some function of x.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$ algebraically



$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(0,1)} = -\frac{0}{1} = 0$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$$

$$\left.\frac{dy}{dx}\right|_{(1,0)} = -\frac{1}{0} \text{ undefined}$$

(vertical tangent)

e.g. Find $\frac{dy}{dx}$ if $x^3+y^3=18xy$

$$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(18xy)$$

$$3x^2+3y^2\frac{dy}{dx}=18\left(x\frac{dy}{dx}+y\right)$$

Solve for $\frac{dy}{dx}$.

$$3x^2+3y^2\frac{dy}{dx}=18x\frac{dy}{dx}+18y$$

$$3y^2\frac{dy}{dx}-18x\frac{dy}{dx}=18y-3x^2$$

$$\frac{dy}{dx}(3y^2-18x)=18y-3x^2$$

$$\frac{dy}{dx}=\frac{18y-3x^2}{3y^2-18x}=\frac{\cancel{3}(6y-x^2)}{\cancel{3}(y^2-6x)}=\frac{6y-x^2}{y^2-6x}\ //$$

Find $\frac{dy}{dx}$ if $x^2 = \frac{x-y}{x+y}$

$$\frac{d}{dx}(x^2) = \frac{d}{dx}\left(\frac{x-y}{x+y}\right)$$

$$2x = \frac{(x+y)\frac{d}{dx}(x-y) - (x-y)\frac{d}{dx}(x+y)}{(x+y)^2}$$

$$2x = \frac{(x+y)(1-\frac{dy}{dx}) - (x-y)(1+\frac{dy}{dx})}{(x+y)^2}$$

$$2x(x+y)^2 = \cancel{x} - x\frac{dy}{dx} + y - \cancel{y\frac{dy}{dx}} - \cancel{x} - x\frac{dy}{dx} + y + \cancel{y\frac{dy}{dx}}$$

$$\cancel{2}x(x+y)^2 = -\cancel{2}x\frac{dy}{dx} + \cancel{2}y$$

$$x(x+y)^2 - y = -x\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x(x+y)^2 - y}{-x} = \frac{y - x(x+y)^2}{x}$$

e.g.

Find equation of tangent line to curve given by $(x^2+y^2)^2 = (x-y)^2$ at $(1,0)$ and $(1,-1)$

$$\frac{d}{dx}(x^2+y^2)^2 = \frac{d}{dx}(x-y)^2$$

$$2(x^2+y^2)\frac{d}{dx}(x^2+y^2) = 2(x-y)\frac{d}{dx}(x-y)$$

$$\cancel{2}(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = \cancel{2}(x-y)\left(1-\frac{dy}{dx}\right)$$

$\underline{(1,0)}$: $(1+0)(2+0) = (1-0)\left(1-\frac{dy}{dx}\right)$

$$2 = 1-\frac{dy}{dx} \qquad \underline{\frac{dy}{dx} = -1}$$

Eqn of tangent line: $y-0 = -1(x-1)$

$$y = -x+1$$

$\underline{(1,-1)}$: $(\cancel{2})\left(2-2\frac{dy}{dx}\right) = (\cancel{2})\left(1-\frac{dy}{dx}\right)$

$$2-2\frac{dy}{dx} = 1-\frac{dy}{dx}$$

$$-\frac{dy}{dx} = -1 \qquad \frac{dy}{dx} = 1$$

Eqn of tangent line:

$$y+1 = 1(x-1)$$

$$y = x-2$$

e.g. Find $\frac{d^2y}{dx^2}$ if $x+\sin(y)=xy$

$$\frac{d}{dx}(x+\sin(y)) = \frac{d}{dx}(xy)$$

$$1+\cos(y)\frac{dy}{dx} = x\frac{dy}{dx}+y$$

$$\frac{d}{dx}\left(1+\cos(y)\frac{dy}{dx}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}+y\right)$$

$$\cos(y)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\left(-\sin(y)\frac{dy}{dx}\right)$$

$$= x\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}(\cos(y)-x) = 2\frac{dy}{dx}+\sin(y)\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2} = \frac{2\frac{dy}{dx}+\sin(y)\left(\frac{dy}{dx}\right)^2}{\cos(y)-x}$$

// This is ok

3.7 Derivatives of Inverse Functions + Logs.

1.6 Inverse Functions

Idea: $y = f(x)$

Each x produces exactly one y.

We often need to solve for x given y.

e.g. $y = 3x + 2$

Given $y = 5$, find x

$$5 = 3x + 2$$
$$3x = 3$$
$$x = 1 \qquad \text{one solution}$$

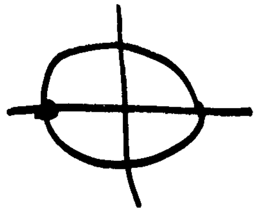
e.g. $y = x^2 + 2x$

Given $y = 0$, find x.

$$x^2 + 2x = 0$$
$$x(x+2) = 0$$
$$x = 0, \; x = -2 \qquad \text{two solutions}$$

eg. $y = \cos(x)$

Given $y = -1$, find x



$\cos(x) = -1$

$x = \pi, 3\pi, 5\pi, 7\pi, \dots$

$-\pi, -3\pi, -5\pi, \dots$

infinitely many solutions

Given $y = f(x)$

If for each y there is <u>only one</u> x such that $y = f(x)$ then we say f is one-to-one and f^{-1} ("f inverse") exists. f^{-1} is a function that gives this solution.

$$\boxed{y = f(x) \iff f^{-1}(y) = x}$$

Usually write $f^{-1}(x)$, not $f^{-1}(y)$.

e.g. $y = 3x + 2 = f(x)$. Find $f^{-1}(x)$.

Solve for x in terms of y:

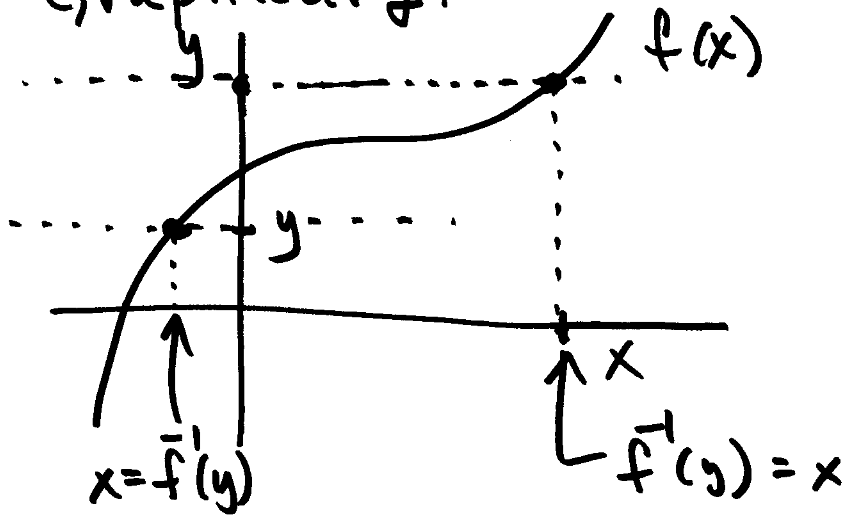
$$y - 2 = 3x$$

$$x = \frac{1}{3}(y-2) \qquad f^{-1}(y) = \frac{1}{3}(y-2)$$

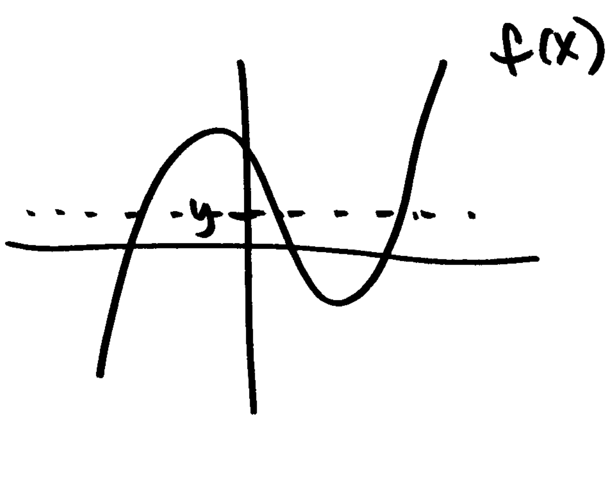
$$\therefore f^{-1}(x) = \frac{1}{3}(x-2)$$

How do I know if f is one-to-one?

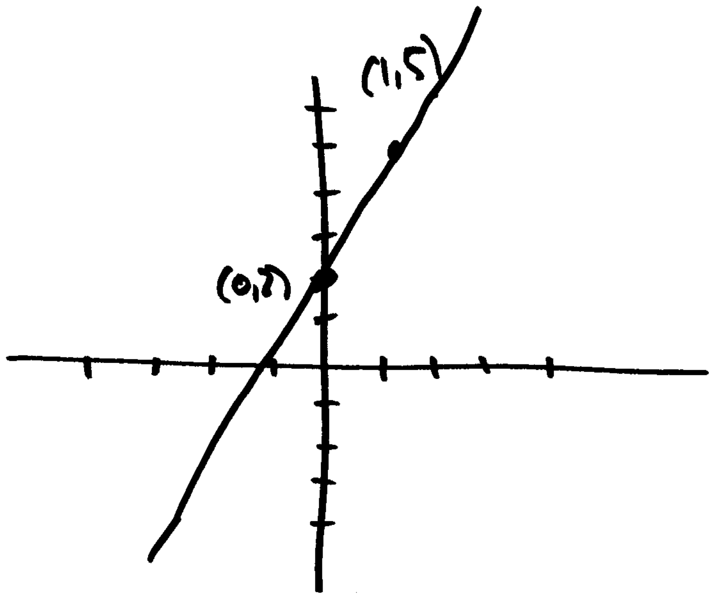
Graphically:



$f(x)$ is one-to-one.

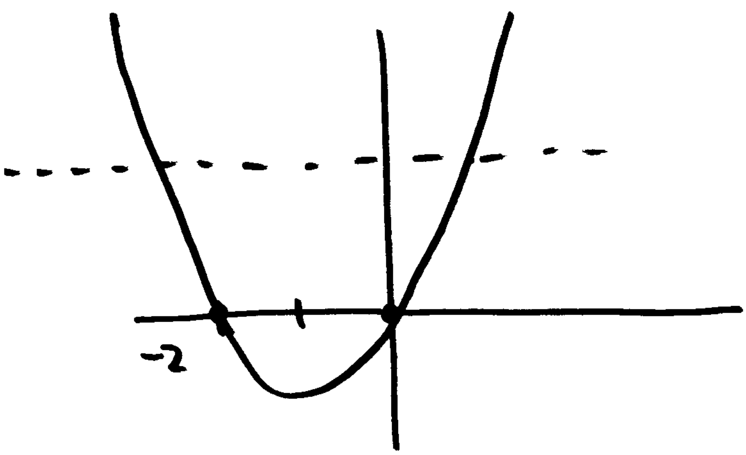


e.g. $y = 3x + 2$



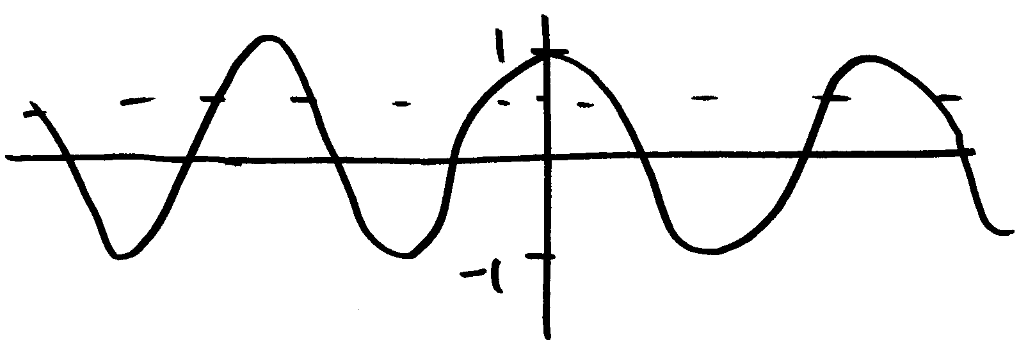
one-to-one

e.g $y = x^2 + 2x$



not one-to-one

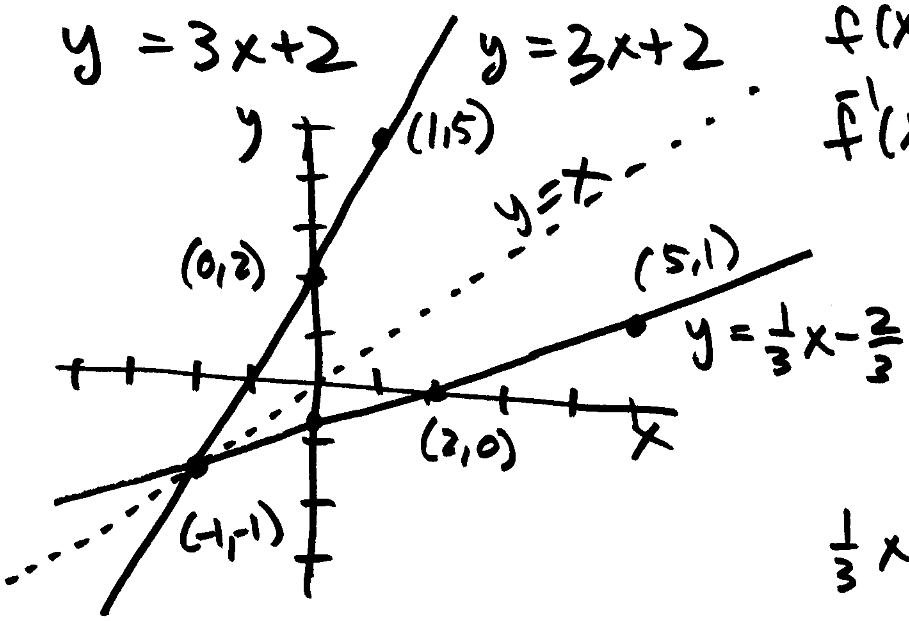
eg $y = \cos(x)$



not one-to-one

Graphs of $f^{-1}(x)$ and $f(x)$

e.g. $y = 3x+2$



$f(x) = 3x+2$

$f^{-1}(x) = \frac{1}{3}(x-2)$

$= \frac{1}{3}x - \frac{2}{3}$

$\frac{1}{3}x - \frac{2}{3} = 3x+2$

$x - 2 = 9x + 6$

$-8 = 8x$

$x = -1$

$y = -1$

$f^{-1}(5) = \frac{1}{3}(5) - \frac{2}{3}$

$= \frac{5}{3} - \frac{2}{3} = \frac{3}{3} = 1$

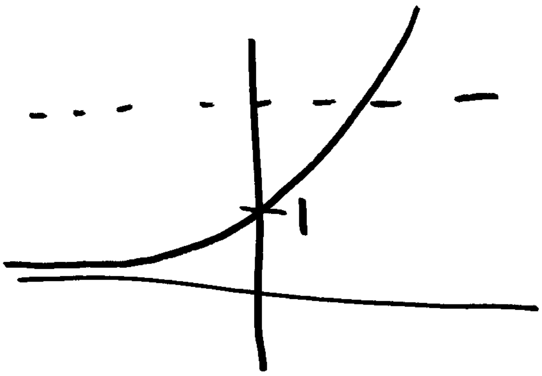
Fact: If (x,y) is on graph of f then (y,x) is on graph of f^{-1}

Fact: $y = f(x) \longleftrightarrow x = f^{-1}(y)$

$$\boxed{\begin{aligned} f(f^{-1}(x)) &= x \\ f^{-1}(f(x)) &= x \end{aligned}}$$

Fundamental example: logarithms

$f(x) = a^x \quad a > 0$, say $a > 1$



Is $f(x)$ one-to-one?

YES

We define $f^{-1}(x) = \log_a(x)$

$$\boxed{y = \log_a(x) \iff a^y = x}$$

Basic facts:

$\log_a(1) = y \iff a^y = 1$

$\boxed{\log_a(1) = 0}$ $\quad y = 0$

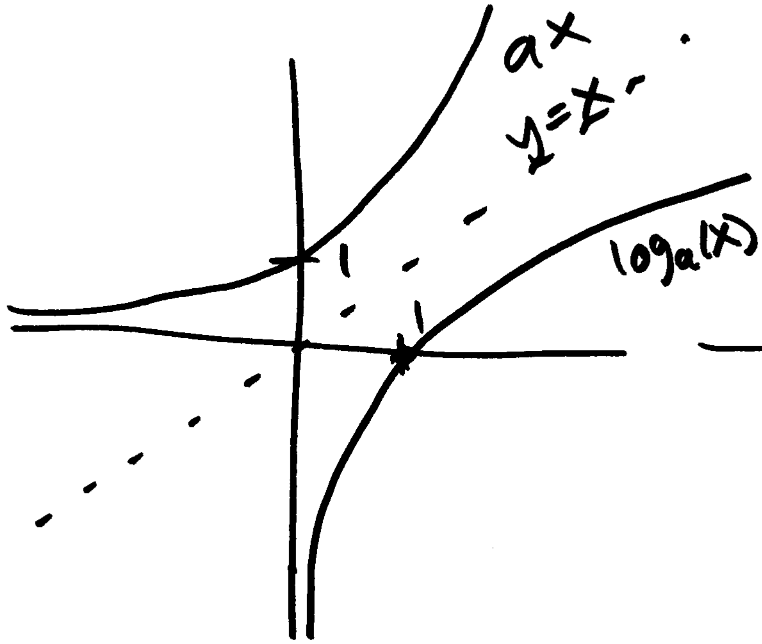
$\boxed{\log_a(a) = 1} \iff a^y = a$

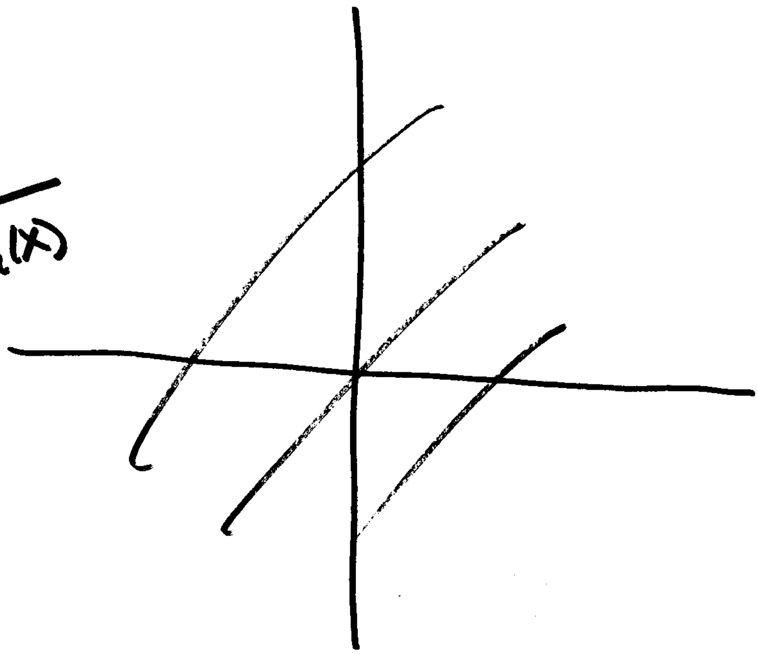
$y = 1$

$\log_a(0) = y \iff a^y = 0$

$\boxed{\log_a(0) \text{ undefined}}$

Graph of $y = \log_a(x)$.





Natural logarithm: $\ln(x) = \log_e(x)$

$\ln(1) = 0$

$\ln(e) = 1$

$\ln(0) =$ undefined

$$e^{\ln(x)} = x$$

$$\ln(e^x) = x$$

Derivatives of inverse functions.

$y = f(x)$ Want $\frac{d}{dx} f^{-1}(x)$.

Use implicit diff.

$$f(f^{-1}(x)) = x$$

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx}(x)$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

e.g. $f^{-1}(x) = \ln(x)$ $f(x) = e^x$

$f'(x) = e^x$

$$\therefore \frac{d}{dx} \ln(x) = \frac{1}{f'(\ln(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}}$$

Exam 2
Cover 3.1–3.6
omit 1.5, 1.6