$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \begin{cases} \text{① slope of tangent line at } x \\ \text{② inst. rate of change of } f \text{ w.r.t } x \text{ \underline{at} } x. \end{cases}$$

Formulas: $\frac{d}{dx}(x^n) = n x^{n-1}$ $\quad n = 0, \pm 1, \pm 2, \dots$

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g); \quad \frac{d}{dx}(cf) = c\frac{d}{dx}(f)$$

Product rule: $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

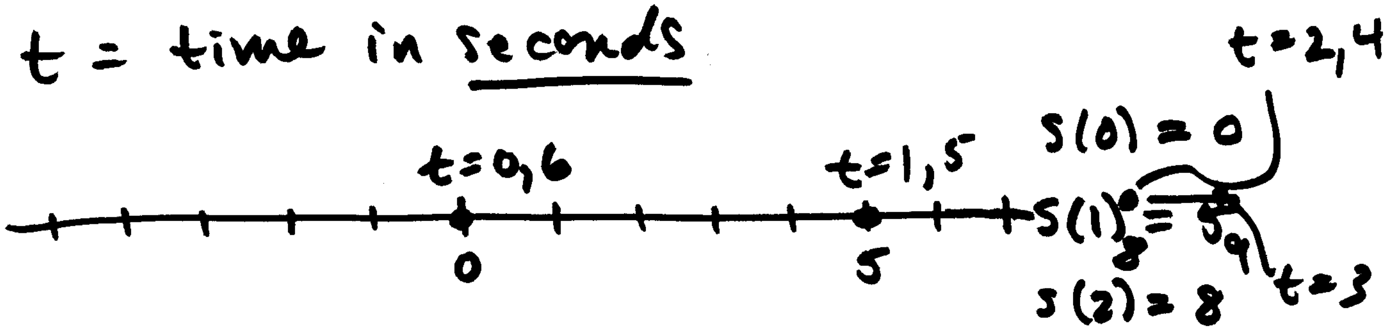
$$\frac{d}{dx}(e^x) = e^x$$

<u>3.3</u> Derivative as a rate of change (cont'd).

<u>eg</u> #2 $\quad s(t) = 6t - t^2 \quad 0 \le t \le 6$

s = position of a body in <u>meters</u>

t = time in <u>seconds</u>



$s(5) = 5$ $\quad$ $s(3) = 9$

$s(6) = 0$ $\quad$ $s(4) = 8$

Total displacement for $0 \le t \le 6$; i.e. the total change in position: $s(6)-s(0)=0$

Average velocity for $0 \le t \le 6$: $\dfrac{s(6)-s(0)}{6-0} = 0$

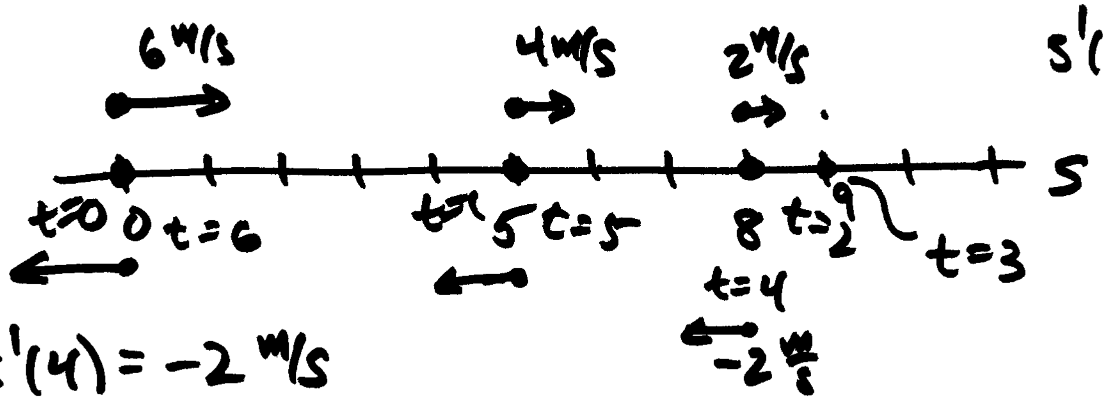
Avg velocity for $0 \le t \le 1$: $\dfrac{s(1)-s(0)}{1-0} = \dfrac{5-0}{1-0} = 5\,\frac{m}{s}$

Instantaneous velocity at any t

is $s'(t) = 6-2t = v(t)$

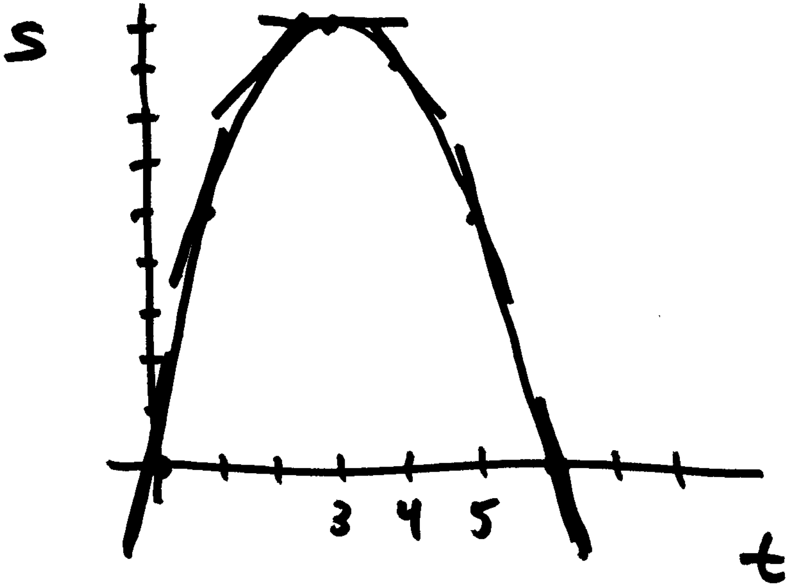
$s'(0) = 6$ m/s. $\quad s'(1) = 4$ m/s $\quad s'(2) = 2$ m/s

$s'(3) = 0$



$s'(4) = -2$ m/s

$s'(5) = -4$ m/s $\quad s'(6) = -6$ m/s.

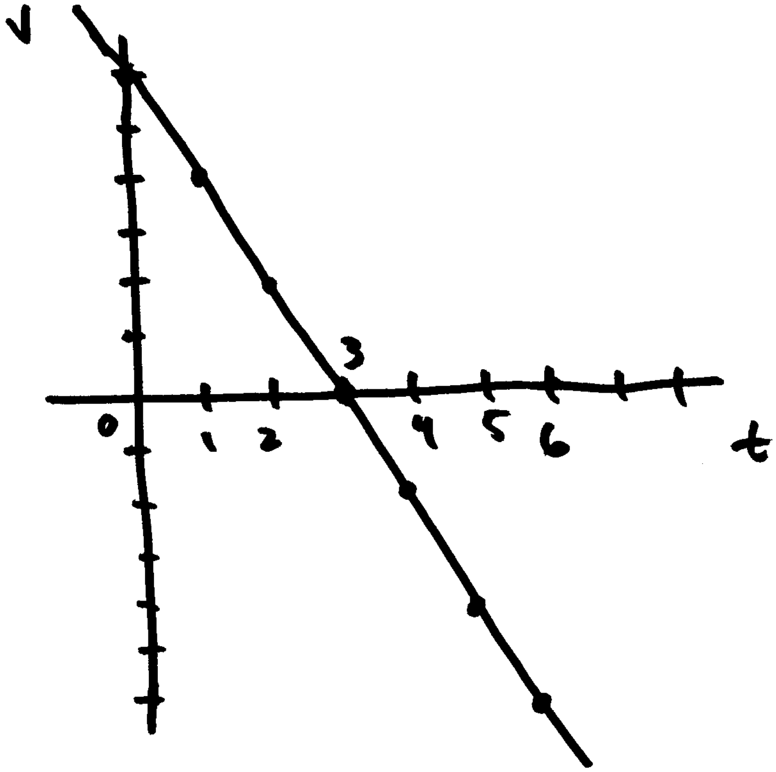


$s = 6t - t^2$

$6t - t^2 = 0$

$t(6 - t) = 0$

$v = s' = 6 - 2t$



acceleration : rate of change of velocity

$s = 6t - t^2$ m

$v = s' = 6 - 2t$ m/s

$a = v' = s'' = -2$ m/s²

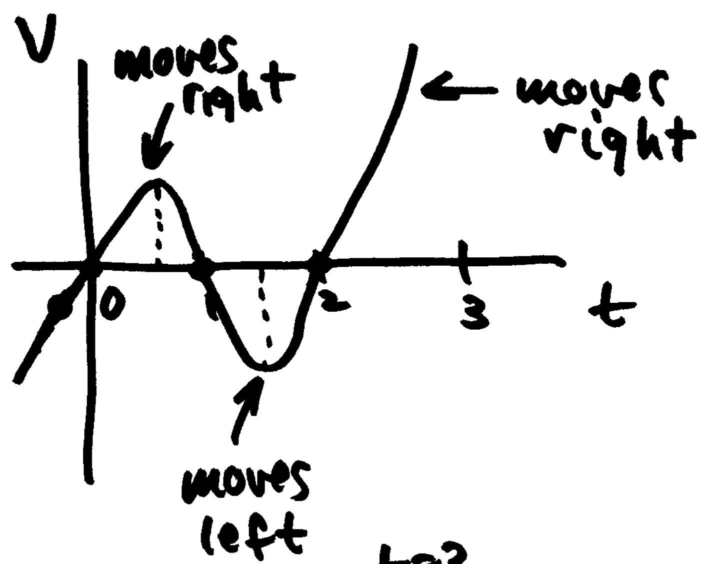
eg #4) $s = \frac{1}{4}t^4 - t^3 + t^2 \quad 0 \le t \le 3$

$$v = s' = \frac{1}{4}4t^3 - 3t^2 + 2t$$
$$= t^3 - 3t^2 + 2t$$

$$a = v' = s'' = 3t^2 - 6t + 2$$

Look at v:

$$v = t^3 - 3t^2 + 2t$$
$$= t(t^2 - 3t + 2)$$
$$= t(t-1)(t-2)$$



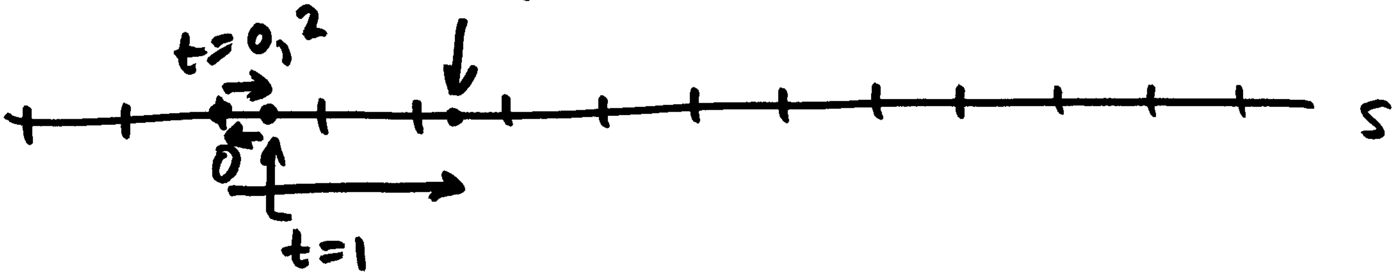
$s(3) = \frac{1}{4} \cdot 81 - 27 + 9$
$= 20.25 - 18 = 2.25$

$s(2) = \frac{1}{4} \cdot 16 - 8 + 4 = 0$

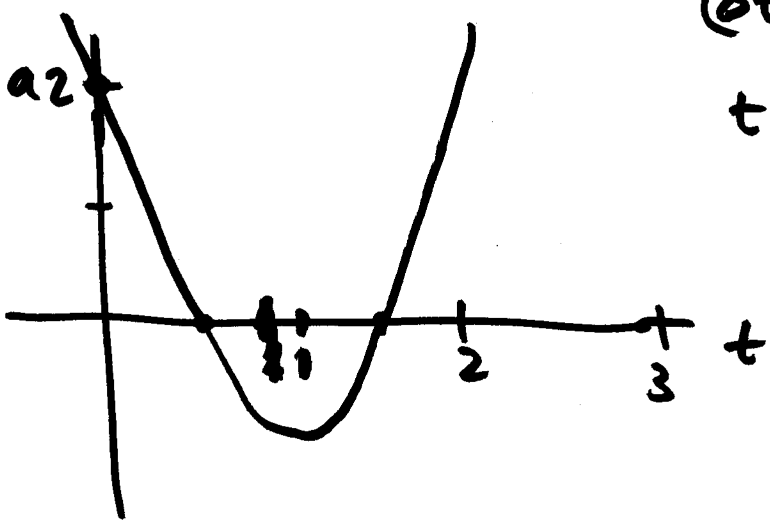
$s(0) = 0$

$s(1) = \frac{1}{4}$

$s = \frac{1}{4}t^4 - t^3 + t^2$



$a = v' = s'' = 3t^2 - 6t + 2$



$t = \dfrac{6 \pm \sqrt{36 - 24}}{6}$

$= \dfrac{6 \pm 2\sqrt{3}}{6}$

$= \dfrac{3 \pm \sqrt{3}}{3}$

$= 1 \pm \dfrac{1}{\sqrt{3}}$

a. total displacement: $s(3) - s(0) = 2.25 - 0 = 2.25$

avg velocity: $\dfrac{s(3) - s(0)}{3 - 0} = \dfrac{2.25}{3} = .75\ \dfrac{m}{s}$

b. $a(0) = 2\ m/s^2$

$a(3) = 3 \cdot 9 - 6 \cdot 3 + 2 = 11\ m/s^2$

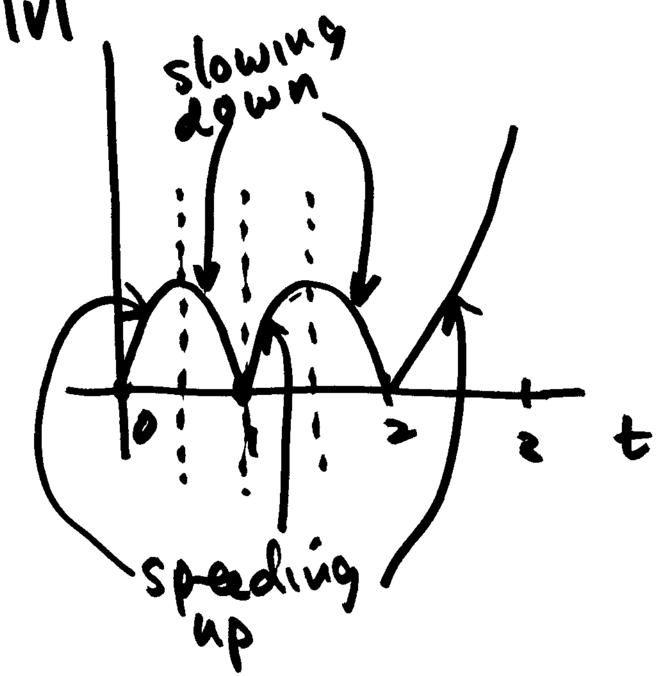
speed: $v(t)$ can be pos and neg.

$|v(t)|$ always positive + is called <u>speed</u> of the body.

$|v(0)| = 0\ m/s \qquad |v(3)| = |27 - 27 + 6| = 6\ m/s.$

c. changes direction at <u>$t=1$</u> and <u>$t=2$</u>.

Look at speed:



3.4 Derivatives of Trig functions

$\sin(x)$ $\qquad\qquad$ $\csc(x) = \dfrac{1}{\sin(x)}$

$\cos(x)$ $\qquad\qquad$ $\sec(x) = \dfrac{1}{\cos(x)}$

$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ $\qquad\qquad$ $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

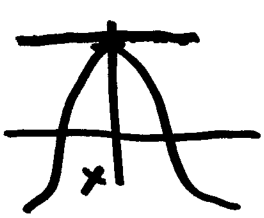
All we need is $\frac{d}{dx}\sin(x)$, $\frac{d}{dx}\cos(x)$.

1. $$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\boxed{\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \sin(x) \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{= \cancel{?}\ 0} + \cos(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{=1}$$



$$\lim_{h\to 0} \frac{\cos(0+h) - \cos(0)}{h} = \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$

slope of tang. line to $\cos(x)$ at $x=0$.

$$= \cos(x)$$

$$\therefore \boxed{\frac{d}{dx} \sin(x) = \cos(x)}$$

2. $$\frac{d}{dx} \cos(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \cos(x) \underbrace{\lim_{h \to 0} \frac{\cos(h) - 1}{h}}_{= 0} - \sin(x) \underbrace{\lim_{h \to 0} \frac{\sin(h)}{h}}_{= 1}$$

$$= -\sin(x)$$

$$\boxed{\frac{d}{dx} \cos(x) = -\sin(x)}$$

3. $\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$

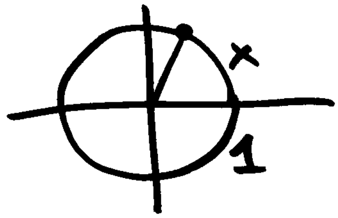
$$= \frac{\cos(x)\frac{d}{dx}\sin(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$x^2+y^2=1$

$(\cos x, \sin x)$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$



$$= \frac{1}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)^2 = \sec^2(x)$$

$$\boxed{\frac{d}{dx}\tan(x) = \sec^2(x)}$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

e.g. $y = x^2 \sin(x)$

$$y' = x^2 \frac{d}{dx}\sin(x) + \sin(x)\cdot\frac{d}{dx}(x^2)$$

$$= x^2\cos(x) + \sin(x)\cdot 2x$$

$$= x^2\cos(x) + 2x\sin(x) \;//$$

eg $y = \dfrac{\cos(x)}{1+\sin(x)}$

$$y' = \frac{(1+\sin(x))\frac{d}{dx}\cos(x) - \cos(x)\frac{d}{dx}(1+\sin(x))}{(1+\sin(x))^2}$$

$$= \frac{(1+\sin(x))(-\sin(x)) - \cos(x)(\cos(x))}{(1+\sin(x))^2}$$

$$= \frac{-\sin(x) - (\sin^2(x) + \cos^2(x))}{(1+\sin(x))^2}$$

$$= \frac{-\sin(x) - 1}{(1+\sin(x))^2} = \frac{-(\cancel{1+\sin(x)})}{(1+\sin(x))^{\cancel{2}}}$$

$$= \frac{-1}{1+\sin(x)}.$$

e.g. $y = (\sin(x) + \cos(x))(\sec(x))$

$$y' = (\sin(x) + \cos(x))(\sec(x)\tan(x))$$
$$+ \sec(x) \cdot (\cos(x) - \sin(x))$$
$$= (\sin(x) + \cos(x))\left(\frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)}\right)$$
$$+ \frac{1}{\cos(x)}(\cos(x) - \sin(x))$$
$$= \left(\frac{\sin(x)}{\cos(x)}\right)^2 + \cancel{\left(\frac{\sin(x)}{\cos(x)}\right)} + 1 - \cancel{\left(\frac{\sin(x)}{\cos(x)}\right)} = \tan(x)^2 + 1$$

Another way:

$$y = (\sin(x) + \cos(x))(\sec(x))$$
$$= (\sin(x) + \cos(x))\left(\frac{1}{\cos(x)}\right)$$
$$= \frac{\sin(x)}{\cos(x)} + 1 = \tan(x) + 1$$

$$y' = \sec^2(x)$$

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\tan^2(x) + 1 = \sec^2(x)$$

e.g. $r = e^{\theta} \tan(\theta)$

$$\frac{dr}{d\theta} = e^{\theta} \sec^2(\theta) + \tan(\theta)\, e^{\theta}$$

$$= e^{\theta}\left(\sec^2(\theta) + \tan(\theta)\right)$$

3.5 Chain rule + Parametric equations

Chain rule.

e.g. $y = f(x) = \sqrt{x^2+1} = (x^2+1)^{1/2}$

Can't find y' with current rules.

If $y = x^2+1$ we could do it

or if $y = \sqrt{x}$ we could do it.

$y = \sqrt{x} \quad y' = \frac{1}{2\sqrt{x}}$

$y = x^{1/2} \quad y' = \frac{1}{2} x^{-\frac{1}{2}}$

Idea: Realize that $y = (x^2+1)^{1/2}$ is a composite function, i.e.

$$y = f(g(x)) \quad \text{where} \quad f(u) = u^{1/2}$$
$$g(x) = x^2+1$$

Also written $y = f \circ g\,(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h)-g(x)}{g(x+h)-g(x)}$$

$$= \underbrace{\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}} \cdot \underbrace{\lim_{h \to 0} \frac{g(x+h)-g(x)}{h}}_{g'(x)}$$

$$\underbrace{\lim_{z \to u} \frac{f(z)-f(u)}{z-u}}_{\text{alternate form for derivative.}}$$

$$= f'(u) = f'(g(x))$$

$$\boxed{\frac{dy}{dx} = f'(g(x)) \cdot g'(x)}$$

In our example $g(x) = x^2+1$

$f(u) = u^{1/2}$

$y = f(g(x)) = (x^2+1)^{1/2}$ $\quad f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$

$y' = f'(g(x)) \cdot g'(x)$ $\quad g'(x) = 2x$

$$= \frac{1}{2\sqrt{g(x)}} \cdot 2x$$

$$= \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$$

Another way to think about it:

$y = f(u) \quad u = g(x) \iff y = f(g(x))$

$$\frac{dy}{dx} = f'(u) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$$

Think of the "du's" as cancelling out.

<u>eg</u> $y = u^{1/2} \quad u = x^2+1$

$\frac{dy}{du} = \frac{1}{2}u^{-1/2} \quad \frac{du}{dx} = 2x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\cancel{2}}u^{-1/2} \cdot \cancel{2}x = x(x^2+1)^{-1/2}$$

eg #4) $y = \cos(u)$ $\quad u = \frac{-x}{3} = -\frac{1}{3}x$

$$\boxed{y = \cos\left(\frac{-x}{3}\right)}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot -\frac{1}{3}$$

$$= \frac{1}{3}\sin(u) = \frac{1}{3}\sin\left(\frac{-x}{3}\right).$$

#6) $y = \sin(u)$ $\quad u = x - \cos(x)$

$$\boxed{y = \sin(x - \cos(x))}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \cdot (1 + \sin(x))$$

$$= \cos(x - \cos(x))(1 + \sin(x))$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}\left[u(x)\right]^n = n\left[u(x)\right]^{n-1}\frac{du}{dx}$$

$y = u^n \quad u = u(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= n\cdot u^{n-1}\cdot\frac{du}{dx}$$

Usually write

$$\boxed{\frac{d}{dx}(u^n) = n u^{n-1}\frac{du}{dx}}$$

e.g. $\frac{d}{dx}\left[(4-3x)^9\right] = 9(4-3x)^8\left(\frac{d}{dx}(4-3x)\right)$

$$= 9(4-3x)^8(-3)$$

$$= -27(4-3x)^8.$$

$\frac{5}{x} = 5x^{-1} \longrightarrow -5x^{-2} = -\frac{5}{x^2}$

$$\frac{d}{dx}\left(\frac{x}{5}+\frac{5}{x}\right)^5 = 5\left(\frac{x}{5}+\frac{5}{x}\right)^4\frac{d}{dx}\left(\frac{x}{5}+\frac{5}{x}\right)$$

$$= 5\left(\frac{x}{5}+\frac{5}{x}\right)^4\left(\frac{1}{5} + -\frac{5}{x^2}\right)$$

$$\boxed{\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}}$$

$$y = e^u \qquad u = u(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}$$

eg $\frac{d}{dx} e^{x^2} = e^{x^2} \cdot 2x$

$$= 2x e^{x^2}$$

$$\frac{d}{dx} e^{\sin(x)} = e^{\sin(x)} \cdot \cos(x)$$

$$= \cos(x) e^{\sin(x)}$$

$$\frac{d}{dx} \sin(e^x) = \cos(e^x) \cdot e^x$$

$$= e^x \cos(e^x)$$

$$\boxed{\frac{d}{dx} \sin(u) = \cos(u) \cdot \frac{du}{dx}}$$

e.g

$$r = (\sec(\theta) + \tan(\theta))^{-1}$$

$$\frac{dr}{d\theta} = -(\sec(\theta) + \tan(\theta))^{-2} \cdot \frac{d}{d\theta}(\sec(\theta) + \tan(\theta))$$

$$= -(\sec(\theta) + \tan(\theta))^{-2}(\sec(\theta)\tan(\theta) + \sec^2(\theta))$$

$$= -\sec(\theta)(\sec(\theta) + \tan(\theta))^{-2}(\tan(\theta) + \sec(\theta))^1$$

$$= -\sec(\theta)(\sec(\theta) + \tan(\theta))^{-1}.$$

e.g.

$$y = (1 + \cot(t/2))^3$$

$$y' = 3(1 + \cot(t/2))^2 \frac{d}{dt}(1 + \cot(t/2))$$

$$= 3(1 + \cot(t/2))^2(-\csc^2(t/2)\frac{d}{dt}(t/2))$$

$$= -3(1 + \cot(t/2))^2 \csc^2(t/2) \cdot \frac{1}{2}$$

$$= \frac{-3}{2}(1 + \cot(t/2))^2 \csc^2(t/2).$$

e.g $\frac{d}{dx}\sin^2(x) = \frac{d}{dx}[\sin(x)]^2 = 2\sin(x) \cdot \cos(x)$

$$\frac{d}{dx}\left(\sin(x^2)\right) = \cos(x^2)\cdot 2x$$

$$= 2x\cos(x^2)$$