Rules for taking derivatives

Quotient rule

$$f(x) = \frac{u(x)}{v(x)} \qquad \frac{d}{dx}(f) = \frac{d}{dx}\left(\frac{u}{v}\right) \neq \cancel{\frac{du/dx}{dv/dx}}$$

NO!! NO!! NO!!

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

eg. $f(t) = \dfrac{2t+5}{t^2-4}$ Find $f'(t)$.

$$f'(t) = \frac{(t^2-4)\frac{d}{dt}(2t+5) - (2t+5)\frac{d}{dt}(t^2-4)}{(t^2-4)^2}$$

$$= \frac{(t^2-4)(2) - (2t+5)(2t)}{(t^2-4)^2}$$

$$= \frac{2t^2 - 8 - 4t^2 - 10t}{(t^2-4)^2}$$

$$= \frac{-2t^2 - 10t - 8}{(t^2-4)^2} \quad //$$

e.g. $p = \dfrac{q^2+3}{q}$ Find $\dfrac{dp}{dq}$

More than one way:

① quotient rule.

$$\frac{dp}{dq} = \frac{q\left(\frac{d}{dq}(q^2+3)\right) - (q^2+3)\left(\frac{d}{dq}(q)\right)}{q^2}$$

$$= \frac{q(2q) - (q^2+3)(1)}{q^2}$$

$$= \frac{2q^2 - q^2 - 3}{q^2}$$

$$= \frac{q^2-3}{q^2} = 1 - \frac{3}{q^2} = 1 - 3q^{-2} \,//$$

② simplify first.

$$p = \frac{q^2+3}{q} = q + \frac{3}{q} = q + 3q^{-1}$$

$$\frac{dp}{dq} = 1 + 3(-1 \cdot q^{-2}) = 1 - 3q^{-2} \,//$$

③ Product rule.

$$p = \frac{q^2+3}{q} = (q^2+3)(q^{-1})$$

$$\frac{dp}{dq} = (q^2+3)\frac{d}{dq}(q^{-1}) + (q^{-1})\frac{d}{dq}(q^2+3)$$

$$= (q^2+3)(-q^{-2}) + (q^{-1})(2q)$$

$$= -1 - 3q^{-2} + 2 = 1 - 3q^{-2} //$$

Higher derivatives

$f(x)$ $f'(x)$ derivative $f''(x)$ second derivative $f'''(x)$ third derivative

etc... $f^{(4)}(x)$ fourth derivative ... $f^{(n)}(x)$ n^{th} derivative

Alternate notation (Leibnitz)

$$y = f(x) \quad \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = \frac{d^2}{dx^2}(f)$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = \frac{d^3f}{dx^3} = \frac{d^3}{dx^3}(f)$$

etc... $$\frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = \frac{d^n}{dx^n}(f)$$

eg #34)
p167 $$u = \frac{(x^2+x)(x^2-x+1)}{x^4}$$

Find u' and u'' $\left(\text{or } \frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}\right)$

Simplify. $$u = \frac{x^4 - \cancel{x^3} + \cancel{x^2} + \cancel{x^3} - \cancel{x^2} + x}{x^4}$$

$$= \frac{x^4+x}{x^4} = 1 + x^{-3}$$

$u' = -3x^{-4}$ //

$u'' = 12x^{-5}$ //

e.g. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x$

Find all derivatives (i.e. of all orders) of f.

$$f'(x) = x^2 + x + \frac{1}{4}$$

$$f''(x) = 2x + 1$$

$$f'''(x) = 2$$

$$f^{(4)}(x) = 0$$

$$f^{(5)}(x) = 0$$

$$\vdots$$

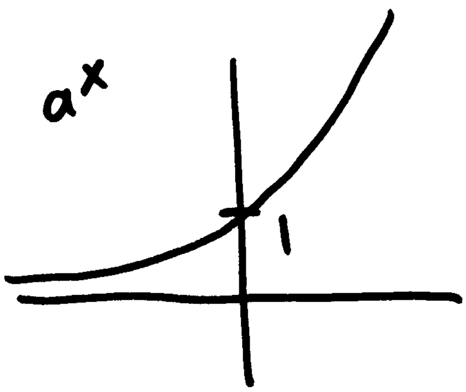
$$f^{(n)}(x) = 0 \quad \text{if } n \geq 4.$$

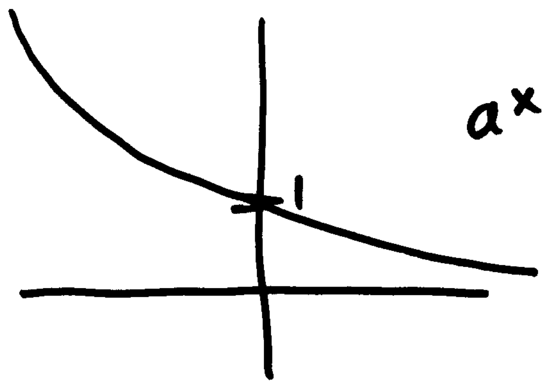
Derivatives of $f(x) = a^x$ $\underline{a>0}$

First discuss $f(x) = e^x$.

$\underline{1.5}$ Exponential functions.

$f(x) = a^x$ $a > 0$



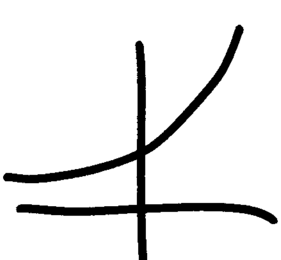


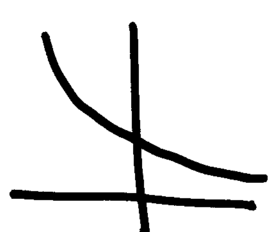
$a > 1$ $0 < a < 1$

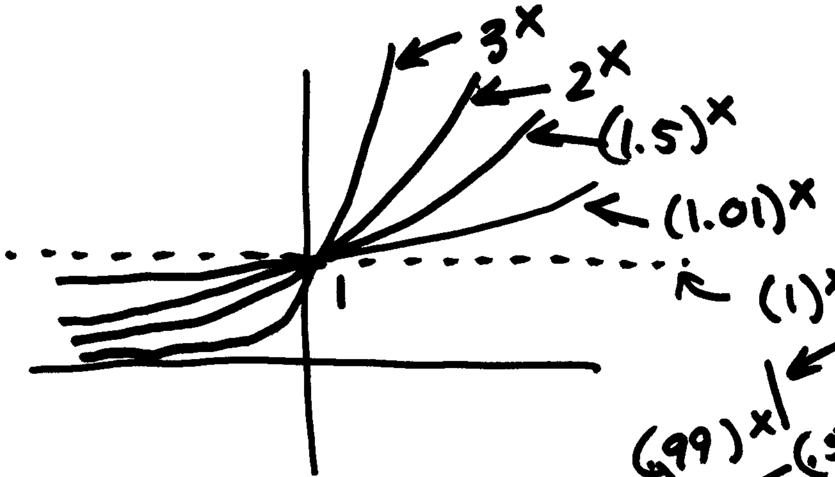
$0 < a < 1 \longrightarrow \frac{1}{a} > 1$

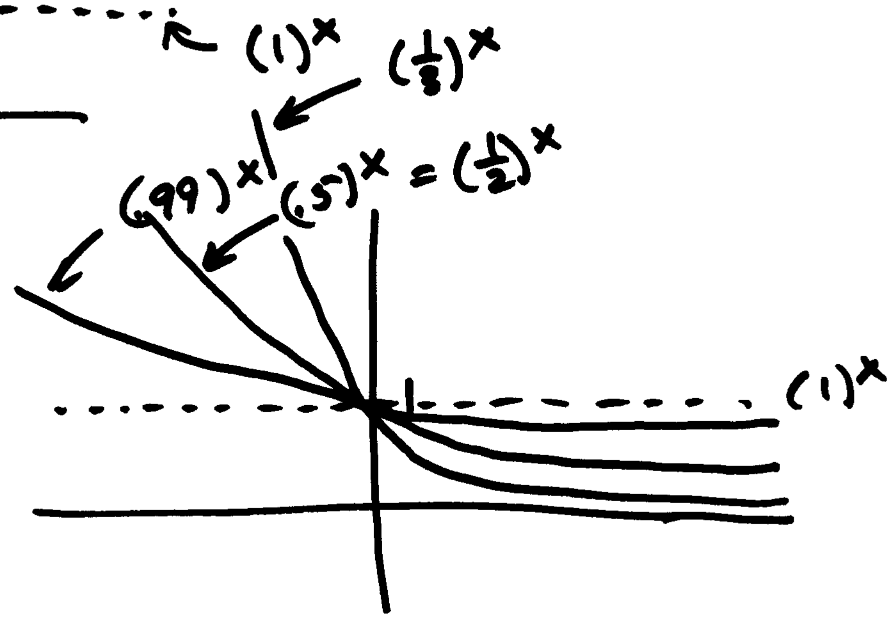
$a^x = \left(\frac{1}{a}\right)^{-x}$

$\underline{\text{e.g.}}$ 2^x $2^{-x} = \left(\frac{1}{2}\right)^x$









<u>useful</u> <u>for</u> <u>modelling</u>

If $a>1$ then $f(x)=a^x$ has a <u>characteristic</u> <u>doubling</u> <u>time</u> T.

That is, for every x $\quad f(x+T) = 2f(x)$

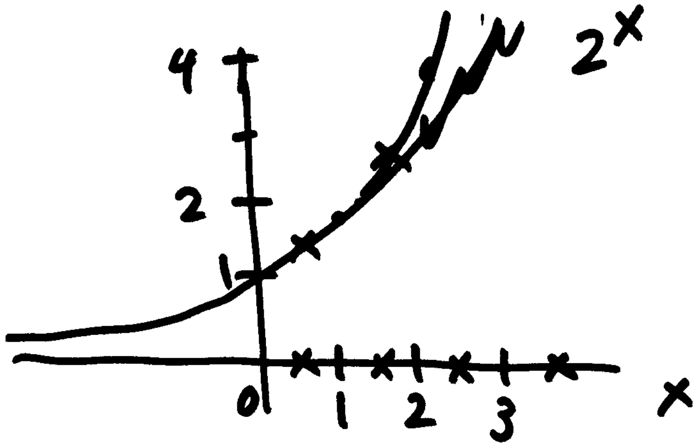
If $0<a<1$ then $f(x)=a^x$ has a <u>characteristic</u> <u>half-life</u> T, that is

$f(x+T) = \frac{1}{2}f(x)$.

<u>e.g.</u> $f(x)=2^x$ then $T=1$

$$f(x+T) = 2^{x+T} = 2^x \cdot 2^T = 2^T f(x)$$

$$f(x+1) = 2f(x)$$



<u>eg</u> $\quad f(x) = (1.5)^x \qquad T = ?$

$$f(x+T) = (1.5)^{x+T} = (1.5)^T (1.5)^x = (1.5)^T f(x)$$

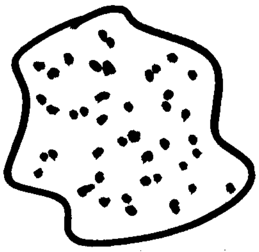
For what T is $(1.5)^T = 2$?

$T = 1$	$T = 2$	$T = 1.5$
$(1.5)^T = 1.5$	$(1.5)^2 = 2.25$	$(1.5)^{1.5} \cong 1.84$

$T = 1.75$
$(1.5)^{1.75} \approx 2.03$

$T = 1.71 \leftarrow$ good approx.
$(1.5)^{1.71} \approx 2.0004.$

e.g. Population models.



Doubling time is "natural"
So exponential models
useful for population growth.

Natural exponential $f(x) = e^x$

example 1, p. 40 \$100 initial deposit
5.5% int compounded annually

A = balance in acct (in \$)

x = # years after 2000

$$A(0) = 100$$
$$A(1) = 100 + 100\cdot(.055) = 100(1.055)$$
$$A(2) = A(1)(1.055) = 100(1.055)^2$$
$$A(3) = A(2)(1.055) = 100(1.055)^3$$
$$\vdots$$
$$A(x) = 100(1.055)^x$$

only accurate if
x = whole number
but is OK approx.

What is doubling time?

$$A(x+T) = 100\,(1.055)^{x+T}$$
$$= 100\,(1.055)^x \cdot (1.055)^T$$
$$= A(x)\,(1.055)^T$$

Solve $(1.055)^T = 2$

$T = 10$ $(1.055)^T \cong 1.71$

$T = 15$ $(1.055)^{15} \approx 2.23$

$T = 13$ $(1.055)^{13} \approx 2.006$

Balance doubles every 13 years.

eg #30) P = population (in # people)

t = # years after 1890

$$P(t) = 6250\,(1.0275)^t$$

pop in 1915 $\longleftrightarrow P(25)$

$$P(25) = 6250\,(1.0275)^{25}$$
$$\approx 6250\,(1.97)$$
$$\cong 12300$$

pop in 1940 $\longleftrightarrow$ $P(50)$

$$P(50) = 6250\,(1.0275)^{50}$$
$$= 6250\left[(1.0275)^{25}\right]^2$$
$$\approx 6250\,\underbrace{(1.97)^2}_{\sim 4}$$
$$\approx 24600$$

When did pop reach 50,000?

Around $t=75$ or 1965

Continuous compounding.

$A(x) = P_0(1+r)^x$ (annual compounding)

P_0 = initial balance

r = interest rate

x = # years since account started.

e.g. in our example

$$A(x) = 100\,(1+.055)^x = 100\,(1.055)^x$$

Compound twice a year.

$$A(x) = 100\,(1.0275)^{2x}$$

$$A(0) = 100$$

$$A(\tfrac{1}{2}) = 100\,(1.0275) = 102.75$$

$$A(1) = 100\,(1.0275)^2$$

$$= (102.75)(1.0275) \cong 105.58$$

$$A(x) = 100\,(1.0275)^{2x}$$

$$= 100\,[(1.0275)^2]^x$$

$$= 100\,(1.05575625)^x$$

$$= 100\left[\left(1 + \frac{.055}{2}\right)^2\right]^x$$

Compound 3x per year

$$A(x) = 100\left[\left(1 + \frac{.055}{3}\right)^3\right]^x$$

$$= 100\,(1.0560144\ldots)^x$$

Compound n times per year.

$$A(x) = 100\left[\left(1+\frac{.055}{n}\right)^n\right]^x$$

In general.

$$A(x) = P_0\left[\left(1+\frac{r}{n}\right)^n\right]^x$$

$$= P_0\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rx}$$

Continuous compounding

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}} = \lim_{h\to 0}(1+h)^{\frac{1}{h}} = e$$

$h = .1 \quad (1.1)^{10} \approx 2.59$

$h = .05 \quad (1.05)^{20} \approx 2.65$

$h = .01 \quad (1.01)^{100} \approx 2.705$

$h = .001 \quad (1.001)^{1000} \approx 2.7169$

$\vdots$

$e = 2.718281828\ldots$

Continuous compounding

$$A(x) = P_0 e^{rt}$$

Get back to derivatives:

$$f(x) = a^x \qquad a > 0$$

Want $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

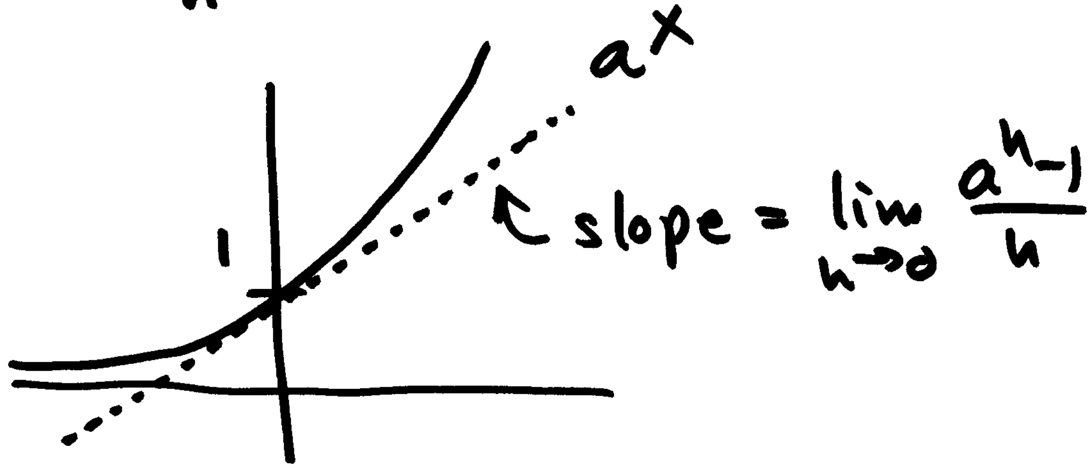
$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= a^x \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{\text{const. depending on } a \text{ not } x.}$$

What is this const?

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$$



For which a is $f'(0) = 1$?

In other words I want

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

$\frac{a^h - 1}{h} \approx 1$ Solve for a:

$$a^h - 1 \approx h$$

$$a^h \approx 1 + h$$

$$a \approx (1+h)^{\frac{1}{h}}$$

$\approx$ becomes $=$ as $h \to 0$

$\therefore\ a = \lim_{h \to 0} (1+h)^{\frac{1}{h}} = e$

Bottom line:

$f(x) = e^x$ then $f'(x) = e^x$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

<u>e.g.</u> $y = x^2 e^x$

Find y' and y''.

$$y' = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$$

$$= x^2 e^x + e^x \cdot 2x$$

$$= (x^2 + 2x) e^x$$

$$y'' = (x^2+2x)\frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2+2x)$$

$$= (x^2+2x)e^x + e^x(2x+2)$$

$$= (x^2+4x+2)e^x$$

e.g. Find $\dfrac{d}{dx}\left(\dfrac{x^2+3e^x}{2e^x-x}\right)$

$e^x \cdot e^x = e^{x+x} = e^{2x}$

$e^x \cdot e^x = (e^x)^2 = e^{2x}$

(quotient rule)

$$= \frac{(2e^x-x)(2x+3e^x)-(x^2+3e^x)(2e^x-1)}{(2e^x-x)^2}$$

$$= \frac{4xe^x + \cancel{6e^{2x}} - 2x^2 - 3xe^x - 2x^2e^x + x^2 - \cancel{6e^{2x}} + 3e^x}{(2e^x-x)^2}$$

$$= \frac{-2x^2e^x + xe^x + 3e^x - x^2}{(2e^x-x)^2} \quad //$$

3.3 Derivative as a rate of change.

$f'(x) =$ instantaneous r.o.c. of $f(x)$ with respect to x at $\underline{x}$.

eg #50 p168

R = blood pressure

$$R = M^2\left(\frac{c}{2} - \frac{M}{3}\right)$$

M = amt. of medicine in blood.

$$\frac{dR}{dM} = \frac{d}{dM}\left(\frac{c}{2}M^2 - \frac{1}{3}M^3\right)$$

R in mm of mercury

$$= \frac{c}{2}\cdot 2M - \frac{1}{3}\cdot 3M^2$$

M in milligrams

$$= CM - M^2$$

units are $\dfrac{\text{mm of mercury}}{\text{milligram}}$

eg 1 p170

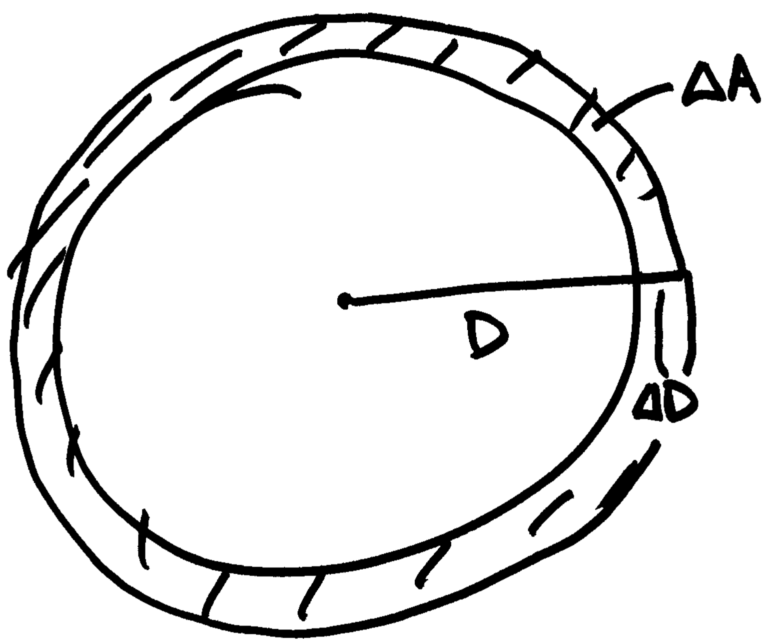
A = area of circle in m^2

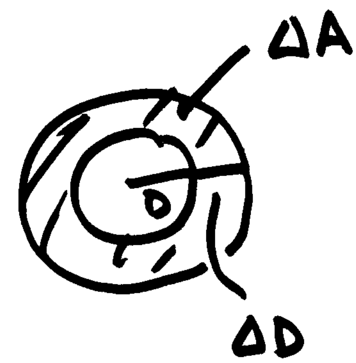
D = diam in m.

~~$D \geq 0$~~

$$A = \frac{\pi}{4} D^2$$

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi}{2} D \ \frac{m^2}{m}$$





$\frac{\Delta A}{\Delta D}$ is bigger if D is big

$\frac{\Delta A}{\Delta D}$ smaller if D small.

$D = 10 \quad \frac{dA}{dD} = 5\pi \approx 15 \ \frac{m^2}{m}$

if $\Delta D = .01$ m then

$\Delta A \approx (5\pi)(.01) \approx .15 \ m^2$