EXAM 1 Friday 5-25

Cover 2.1-2.7 omit 2.3

2.2 47) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f(x)=\sqrt{x}$ $\quad x=7$

$$\lim_{h\to 0}\frac{f(7+h)-f(7)}{h} = \lim_{h\to 0}\frac{\sqrt{7+h}-\sqrt{7}}{h}$$

$$\frac{\sqrt{7+h}-\sqrt{7}}{h}\cdot\frac{\sqrt{7+h}+\sqrt{7}}{\sqrt{7+h}+\sqrt{7}} = \frac{\cancel{7}+h-\cancel{7}}{h(\sqrt{7+h}+\sqrt{7})}$$

$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{7+h}+\sqrt{7})} = \frac{1}{\sqrt{7+h}+\sqrt{7}}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{7+h}+\sqrt{7}} = \frac{1}{2\sqrt{7}}$$

2.2 27) $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1} = \lim\limits_{u\to 1} \dfrac{(u+1)(u^2+1)}{u^2+u+1} = \dfrac{4}{3}$

$$\frac{u^4-1}{u^3-1} = \frac{(u^2-1)(u^2+1)}{(u-1)(u^2+u+1)} = \frac{\cancel{(u-1)}(u+1)(u^2+1)}{\cancel{(u-1)}(u^2+u+1)}$$

$$= \frac{(u+1)(u^2+1)}{u^2+u+1}$$

$$\begin{array}{r|l} & u^2+u+1 \\ \hline u-1 & u^3+0\cdot u^2+0\cdot u-1 \\ & -(u^3-u^2) \\ \hline & u^2+0\cdot u \\ & -(u^2-u) \\ \hline & u-1 \\ & u-1 \\ \hline & 0 \end{array}$$

2.7 13) $g(x) = \dfrac{x}{x-2}$ $\quad \underline{(3,3)}$ point $\quad x=3$

slope of tangent line $= \lim\limits_{h\to 0} \dfrac{g(3+h)-g(3)}{h}$

$$= \lim_{h\to 0} \frac{\frac{3+h}{3+h-2} - 3}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{3+h}{1+h} - 3\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{3+h-3(1+h)}{1+h}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{\cancel{3}+h-\cancel{3}-\cancel{3}h}{1+h}\right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{-2\cancel{h}}{1+h}\right) = \lim_{h\to 0} \frac{-2}{1+h} = -2 \; //$$

Eqn of tangent line:

$$y-3 = -2(x-3)$$

$$y = 3-2x+6$$

$$y = -2x+9 \; //.$$

2.7 9) $y = x^3 = f(x)$ $(-2,-8)$ // ← point

slope: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h}$

$$= \lim_{h\to 0} \frac{(-2+h)^3+8}{h} = \lim_{h\to 0} h^2-6h+12 = 12 \; //$$

$$\frac{(h-2)^3+8}{h} = \frac{h^{\cancel{3}2}-6h^{\cancel{2}}+12\cancel{h}-\cancel{8}\cancel{+8}}{\cancel{h}}$$

$(h-2)^3 = (h-2)(h-2)^2$

$= (h-2)(h^2-4h+4)$

$= h^3-6h^2+12h-8$

Eqn of tangent line:

$$y + 8 = 12(x+2)$$

$$y = -8 + 12x + 24$$

$$y = 12x + 16 \quad //$$

3.1 Derivative as a function.

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$:

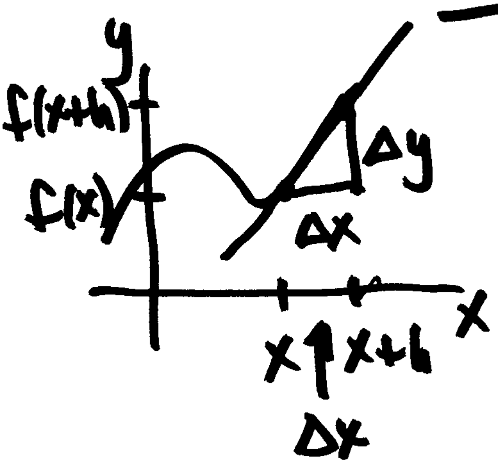
1) slope of tangent line <u>at</u> $x=x_0$
2) instantaneous rate of change of f with respect to x <u>at</u> $x=x_0$.

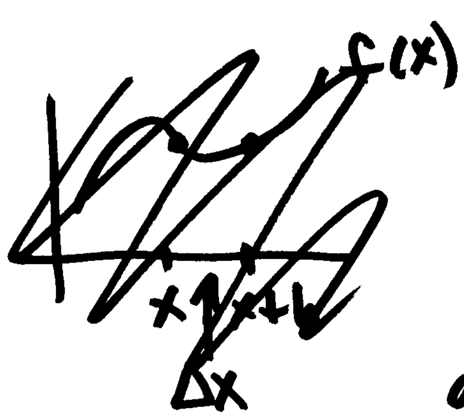
Replace x_0 by the <u>variable</u> x we can define the <u>derivative</u> of $f(x)$, $f'(x)$.

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

<u>Notation</u>: $y=f(x)$

<u>$f'(x)$</u> $\quad \underline{\dfrac{dy}{dx}}=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$





$$\underline{\frac{df}{dx}}$$

$$\underline{\frac{d}{dx}(f)}$$

e.g. $f(x) = x^2 - 2x + 2$

Find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) + 2 - (x^2 - 2x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h + \cancel{2} - \cancel{x^2} + \cancel{2x} - \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h - 2)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h - 2 = 2x - 2$$

$f'(x) = 2x - 2$

Find slope of tangent line at $x = -1$:

$f'(-1) = -2 - 2 = -4$ //

at $x = 2$: $f'(2) = 2$ //

at $x = 1$: $f'(1) = 0$ //

at $x = 0$: $f'(0) = -2$ //

e.g. $f(x) = \sqrt{2x+1}$ Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$$

$$\frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

$$= \frac{2(x+h)+1-(2x+1)}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \frac{\cancel{2x} + 2\cancel{h} \cancel{+1} \cancel{-2x} \cancel{-1}}{\cancel{h}\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \frac{2}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

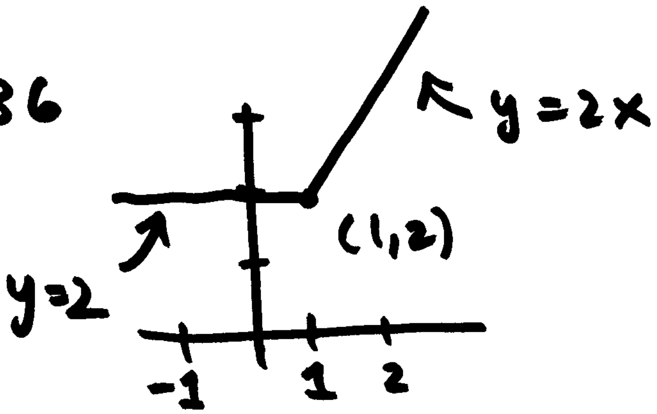
$$\lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+1}+\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

$$\therefore\ f'(x) = \frac{1}{\sqrt{2x+1}}$$

Failure of $f'(x)$ to exist.

① One-sided limits differ. (corner)

#36



$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ 2 & \text{if } x < 1 \end{cases}$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$h>0$, $1+h>1$ so $f(1+h) = 2(1+h)$

$$\therefore\ \frac{f(1+h)-f(1)}{h} = \frac{2(1+h)-2}{h} = \frac{\cancel{2}+2\cancel{h}-\cancel{2}}{\cancel{h}} = 2$$

$$\therefore \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} 2 = 2 \,//$$

$h<0 \quad 1+h<1 \quad$ so $\quad f(1+h) = 2$

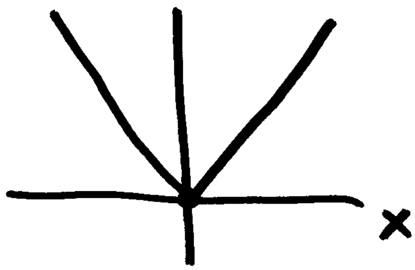
$$\therefore \frac{f(1+h)-f(1)}{h} = \frac{2-2}{h} = 0$$

$$\therefore \lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^-} 0 = 0 \,//$$

$$\therefore \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$$ <u>does not exist.</u>

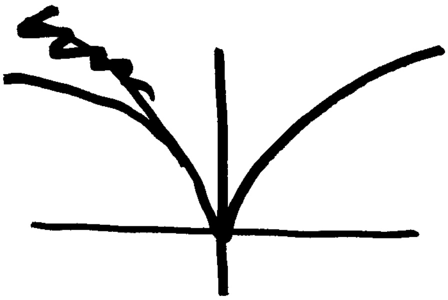
<u>e.g.</u> $f(x) = |x|$

$f'(0)$ does not exist.



② infinite limit (cusps or vertical tangent)

<u>eg</u> $f(x) = x^{2/3}$ Does $f'(0)$ exist?

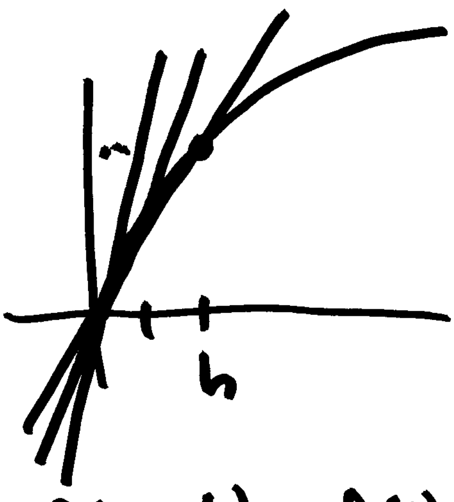


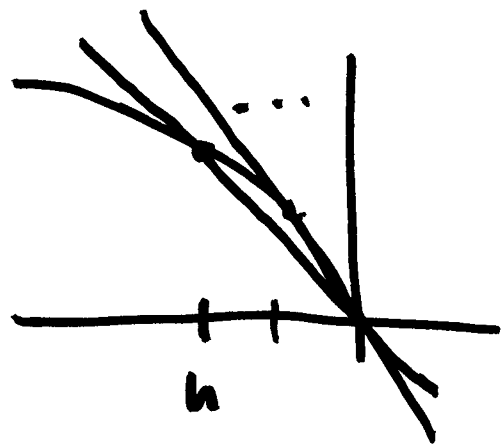
<u>NO</u>

look at one-sided derivatives <u>at $x=0$</u>

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$$

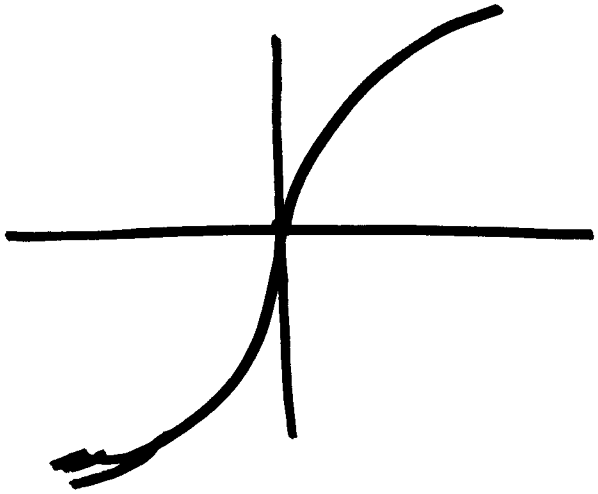
$$= \lim_{h\to 0^+} \frac{h^{2/3}}{h} = \lim_{h\to 0^+} \frac{1}{h^{1/3}} = +\infty$$





$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{1}{h^{1/3}} = -\infty$$

e.g. $f(x) = x^{1/3}$ (odd function)



$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$$

vertical tangent

REVIEW

2.4

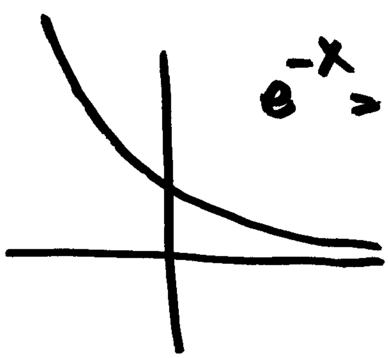
45) $$\lim_{t\to -\infty} \frac{2-t+\sin t}{t+\cos t}$$

$$= \lim_{t\to-\infty} \frac{-t}{t} = \lim_{t\to-\infty} -1 = -1$$

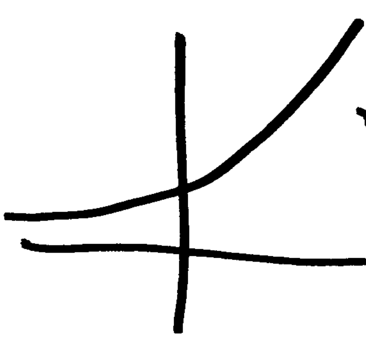
$$\lim_{t\to-\infty} \frac{2-t+\sin t}{t+\cos t}\cdot\frac{\frac{1}{t}}{\frac{1}{t}}$$

$$= \lim_{t\to-\infty} \frac{\frac{2}{t}-1+\frac{\sin t}{t}}{1+\frac{\cos t}{t}} = \frac{0-1+0}{1+0} = -1$$

47) $$\lim_{x\to\infty} \underbrace{e^{-x}}_{\downarrow\, 0}\,\sin(x) = 0$$

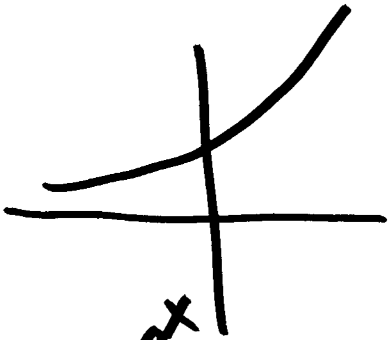


$e^{-x} = \frac{1}{e^x}$

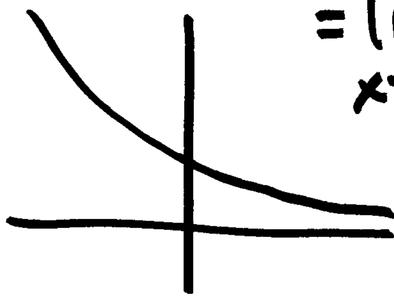


e^x

49) $$\lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to-\infty} \frac{-e^{-x}}{e^{-x}}$$



e^x



e^{-x}

$$= \lim_{x\to-\infty} -1 = -1.$$

$f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ has a horiz asymptote at $y = -1$ and $y = 1$

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{e^x}{e^x} = 1$$

2.6 (13–19)

13) $y = \frac{1}{x-2} - 3x$ continuous for all $x \neq 2$

$\lim_{x \to c} f(x) = f(c) \iff f(x)$ is continuous at $x = c$.

$y =$ polynomial

eg $y = x^3 + 3x^2 - x + 1$

cont for all x, or for x in $(-\infty, \infty)$

$y =$ rational ftn $= \frac{\text{polynomial}}{\text{polynomial}}$

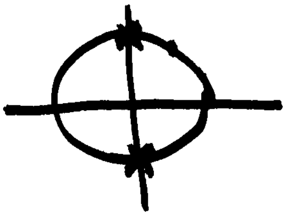
eg. $y = \frac{x+1}{x^2 - 4x + 3}$

continuous at every x for which denominator $\neq 0$, i.e. for every $x \neq 1$ and $\neq 3$.

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0 \quad x = 1, x = 3$

e.g. $y = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

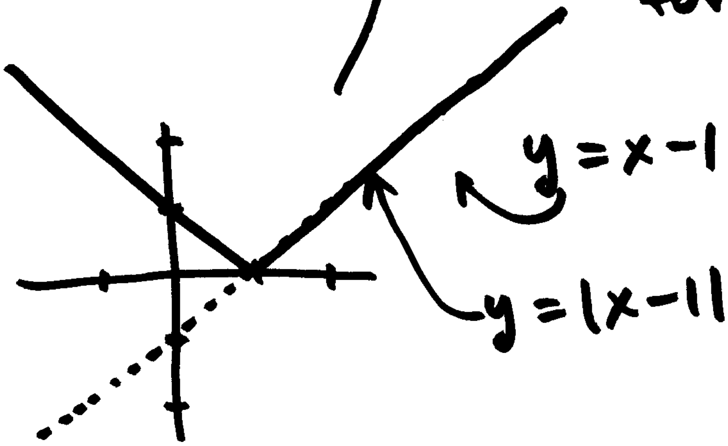


continuous for all x

except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

or $x = \frac{\pi}{2} + n\pi \quad n = 0, \pm 1, \pm 2, \ldots$

17) $y = \overbrace{|x-1|}^{\text{continuous for all } x} + \underbrace{\sin(x)}_{\text{continuous for all } x}$



3.2 Differentiation Rules

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Polynomials

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

eg $\quad y = 1 + 3x + x^2$

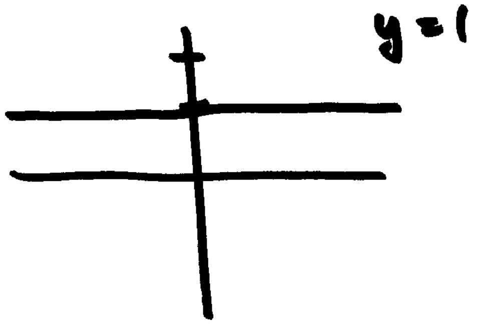
$\quad\quad y = \frac{4}{3}x^4 + 5x^2$

$\vdots$

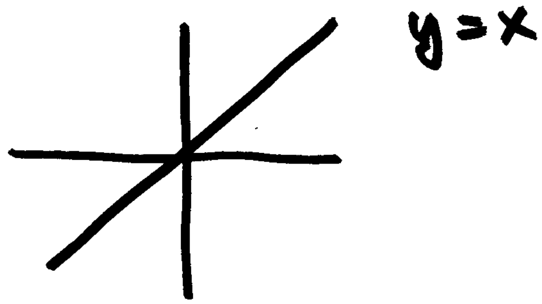
a. Derivative of x^n

$n = 0: \quad x^n = x^0 = 1$

$$\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0$$



$n=1: \quad \frac{d}{dx}(x^n) = \frac{d}{dx}(x) = 1$



$$n=2: \quad \frac{d}{dx}(x^2) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2x\cancel{h}+h^2-\cancel{x^2}}{\cancel{h}} = \lim_{h\to 0} 2x+h = 2x$$

$$\frac{d}{dx}(x^2) = 2x$$

$$n=3: \quad \frac{d}{dx}(x^3) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2\cancel{h}+3xh^2+h^3-\cancel{x^3}}{\cancel{h}} = \lim_{h\to 0} 3x^2+3xh+h^2$$

$$= 3x^2$$

$$(x+h)(x+h)^2 = (x+h)(x^2+2xh+h^2)$$
$$= x^3+3x^2h+3xh^2+h^3$$

$$\frac{d}{dx}(x^3) = 3x^2$$

In general

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1} \quad \text{for } n = 0, 1, 2, \ldots}$$

Explain:

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$(x+h)^n = x^n + n x^{n-1} h + (\ \ ) h^2 + (\ \ ) h^3 + \cdots + (\ \ ) h^n$$

1	$(x+h)^0 = 1$
$1 \quad 1$	$(x+h)^1 = x + h$
$1 \quad 2 \quad 1$	$(x+h)^2 = x^2 + 2xh + h^2$
$1 \quad 3 \quad 3 \quad 1$	$(x+h)^3 = x^3 + 3x^2 h + 3xh^2 + h^3$
$1 \quad 4 \quad 6 \quad 4 \quad 1$	$(x+h)^4 = x^4 + 4x^3 h + 6x^2 h^2 + 4xh^3 + h^4$
$\vdots$	$\vdots$

$$= \lim_{h \to 0} n x^{n-1} + \underbrace{(\ )h + (\ )h^2 + \cdots (\ )h^{n-1}}_{\to 0}$$

$$= n x^{n-1}$$

b. Derivative is "linear"

<u>means</u>: $\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

and $\frac{d}{dx}\left(c\, f(x)\right) = c \frac{d}{dx} f(x)$

<u>e.g.</u> $\frac{d}{dx}(x^4) = 4x^3$

$$\frac{d}{dx}(5x^4) = 5 \frac{d}{dx}(x^4) = 5 \cdot 4x^3 = 20x^3$$

$$\frac{d}{dx}(x^4 + x^3) = \frac{d}{dx}(x^4) + \frac{d}{dx}(x^3)$$
$$= 4x^3 + 3x^2$$

$$\frac{d}{dx}(3x^4 + 5x^3 + 2)$$
$$= 3\frac{d}{dx}(x^4) + 5\frac{d}{dx}(x^3) + 2\frac{d}{dx}(1)$$
$$= 12x^3 + 15x^2 + 2 \cdot 0 = 12x^3 + 15x^2.$$

Product rule: $f(x) = u(x) \cdot v(x)$

$$\frac{d}{dx}(u \cdot v) = \cancel{\frac{d}{dx}(u) \cdot \frac{d}{dx}(v)}$$

No!! No!!

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{d}{dx}(v) + v \cdot \frac{d}{dx}(u)$$

Why?

$$\frac{d}{dx}(u \cdot v) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h)[v(x+h) - v(x)] + v(x)[u(x+h) - u(x)]}{h}$$

$$= \lim_{h \to 0} u(x+h)\left(\frac{v(x+h) - v(x)}{h}\right) + \lim_{h \to 0} v(x)\left(\frac{u(x+h) - u(x)}{h}\right)$$

$$= u(x)\, v'(x) + v(x)\, u'(x)$$

<u>e.g.</u> $\frac{d}{dx}\left[(x^3+x+2)(x^4+2x^2)\right]$

$$= (x^3+x+2)\frac{d}{dx}(x^4+2x^2)$$
$$\qquad + (x^4+2x^2)\frac{d}{dx}(x^3+x+2)$$

$$= (x^3+x+2)(4x^3+4x) + (x^4+2x^2)(3x^2+1)$$

$$= 4x^6 + 4x^4 + 4x^4 + 4x^2 + 8x^3 + 8x$$
$$\qquad + 3x^6 + x^4 + 6x^4 + 2x^2$$

$$= 7x^6 + 15x^4 + 8x^3 + 6x^2 + 8x$$

Another way:

$$\frac{d}{dx}\left[(x^3+x+2)(x^4+2x^2)\right]$$

$$= \frac{d}{dx}\left[x^7 + 2x^5 + x^5 + 2x^3 + 2x^4 + 4x^2\right]$$

$$= \frac{d}{dx}\left[x^7 + 3x^5 + 2x^4 + 2x^3 + 4x^2\right]$$

$$= 7x^6 + 15x^4 + 8x^3 + 6x^2 + 8x$$

One more thing:

Spse $n = 0, 1, 2, 3, \ldots$

$$x^n \cdot \frac{1}{x^n} = 1$$

$$\frac{x^{n-1}}{x^n} = x^{n-1-n} = x^{-1}$$

$$\frac{d}{dx}\left(x^n \cdot \frac{1}{x^n}\right) = \frac{d}{dx}(1) = 0$$

$$x^n \frac{d}{dx}\left(\frac{1}{x^n}\right) + \frac{1}{x^n} \cdot (n x^{n-1}) = 0$$

$$x^n \underbrace{\frac{d}{dx}\left(\frac{1}{x^n}\right)} + n x^{-1} = 0$$

$$\therefore \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n \cdot x^{-1}}{x^n} = -n x^{-n-1}$$

$$\frac{d}{dx}(x^{-n}) = -n x^{-n-1}$$

$$\therefore \boxed{\frac{d}{dx}(x^n) = n x^{n-1} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots}$$