2.6 Continuity

Idea: We know that sometimes

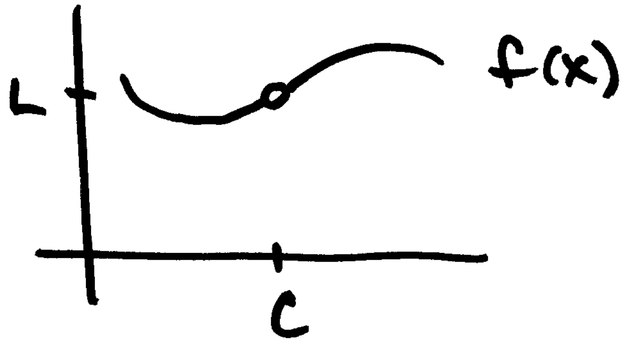
$$\lim_{x \to c} f(x) = f(c)$$

Def: A function $f(x)$ is <u>continuous</u> at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

Note that 3 things must happen:

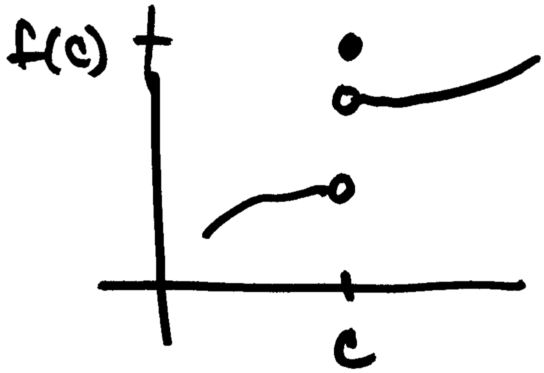
1. $f(c)$ must exist.
2. $\lim_{x \to c} f(x)$ must exist.
3. both must be equal.

Any of these may fail.



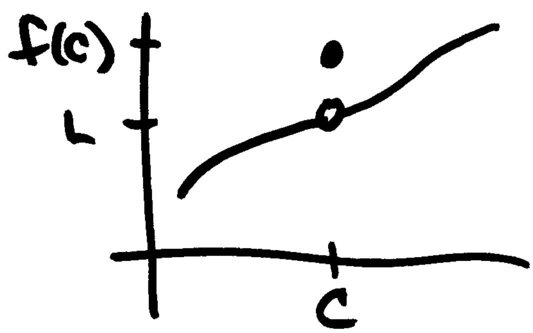
② holds but ① fails

so f is <u>not</u> continuous at $x=c$



① holds but ② fails.

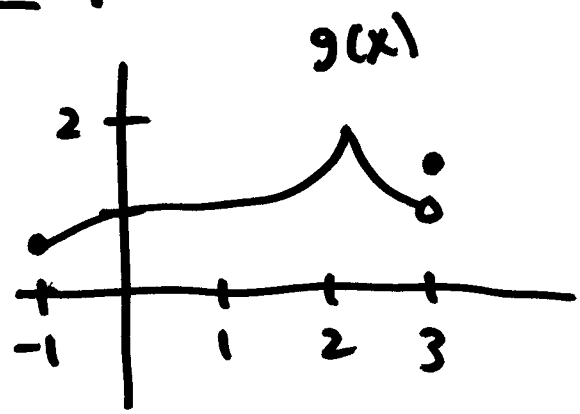
so f not continuous at $x=c$



① + ② hold but ③ fails.

so f is not continuous at $x=c$.

<u>eg #2</u> p129



Continuous on $[-1,3]$?

Not continuous at $x=3$ because

$$\lim_{x\to 3^-} g(x) \neq g(3)$$

$g(x)$ is continuous at $x=-1$

since $\lim_{x \to -1^+} g(x) = g(-1)$

eg #6

eg $f(x) = \dfrac{x^2-1}{x-1}$

Is $f(x)$ cont. at $x=1$? No. $f(1)$ is not defined.

Is the discontinuity removable?

Yes. $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = \lim_{x \to 1} \dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$

$= \lim_{x\to 1} x+1 = 2$

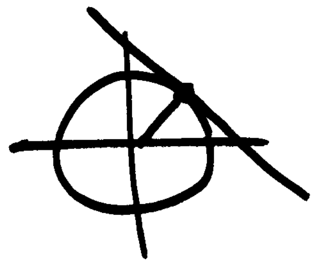
If we define $f(1)=2$ then f is continuous.

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$ is continuous.

2.7 Tangents + Derivatives

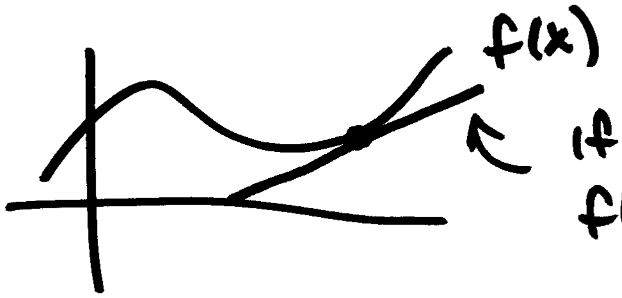
Tangents

- circle



tangent line:
- touches circle at only one pt.
- perpendicular to radius
- if moving on circle will fly off on tangent line.

- in general



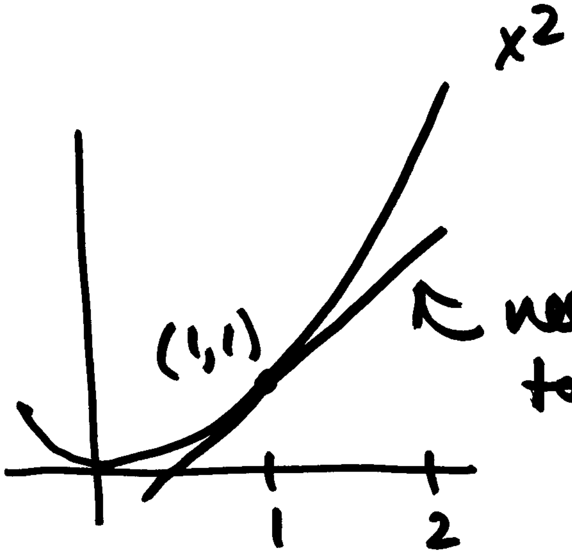
if released, would fly off along this line.

Moving along curve consists of a ~~series~~ continuous series of linear motions.

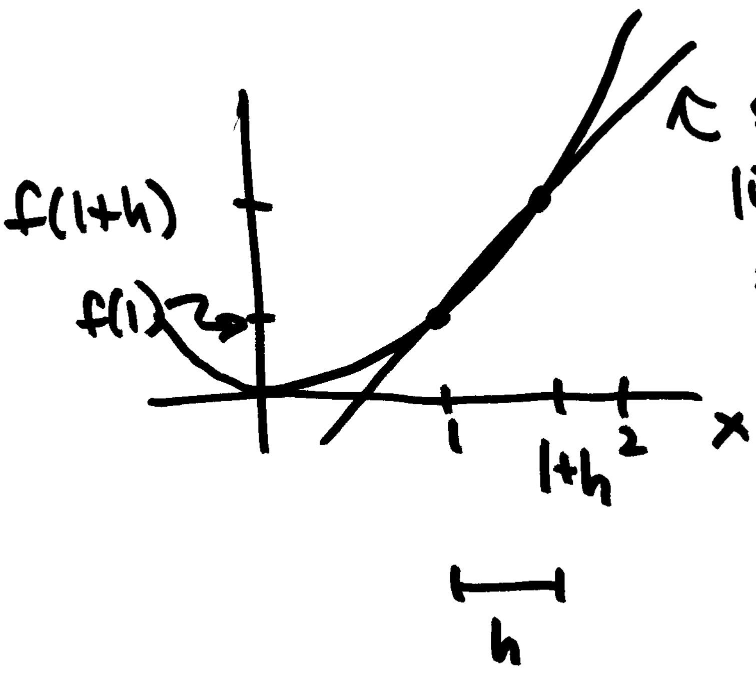
These lines are called tangent lines to the curve.

e.g. $f(x) = x^2$

Find equation of tangent line at $x = 1$



need slope of tangent line.



slope of secant line

$$= \frac{f(1+h) - f(1)}{h}$$

= average r.o.c. of $f(x)$ on $[1, 1+h]$.

slope of tangent line $= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

= instantaneous r.o.c. of $f(x)$ at $x = 1$.

In our example.

slope of tangent line $= \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$

$$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = \lim_{h \to 0} \frac{2h + h^2}{h}$$

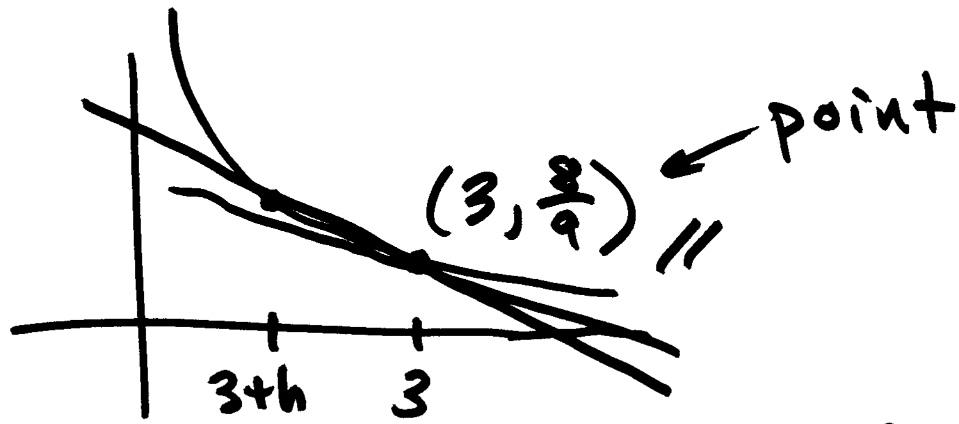
$$= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h \to 0} 2 + h = 2$$ //

$\therefore$ eqn of tangent line: $\boxed{y - y_0 = m(x - x_0)}$

$$y - 1 = 2(x - 1)$$
$$y = 1 + 2(x - 1)$$
$$y = 2x - 1$$ //

eg Find eqn of tangent line to

$g(t) = \frac{8}{t^2}$ at $t = 3$



$$\text{slope of tangent line} = \lim_{h \to 0} \frac{g(3+h) - g(3)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{8}{(3+h)^2} - \frac{8}{9}}{h} = \lim_{h \to 0} \frac{-48-8h}{9(3+h)^2} = \frac{-48}{81}$$

$$= \frac{-16}{27} //$$

$$\frac{1}{h}\left(\frac{8}{(3+h)^2} - \frac{8}{9}\right) = \frac{1}{h} \frac{72 - 8(3+h)^2}{9(3+h)^2}$$

$$= \frac{72 - 8(9+6h+h^2)}{9h(3+h)^2}$$

$$\boxed{\frac{-48h-8h^2}{9h(3+h)^2} = \frac{h(-48-8h)}{9h(3+h)^2}}$$

$$= \frac{\cancel{72} - \cancel{72} - 48\cancel{h} - 8h^2}{9\cancel{h}(3+h)^2} = \frac{-48-8h}{9(3+h)^2}$$

Eqn of tangent line:

$\left(3, \frac{8}{9}\right)$ slope $= \frac{-16}{27}$

$$y - \frac{8}{9} = -\frac{16}{27}(x-3)$$

$$y = \frac{8}{9} - \frac{16}{27}x + \frac{16}{9}$$

$$y = -\frac{16}{27}x + \frac{24}{9} = -\frac{16}{27}x + \frac{8}{3} \quad //$$

<u>e.g.</u> Find eqn of tangent line to $f(x) = \frac{x-1}{x+1}$ at $x=0$

point: $(0,-1)$ //

slope: $$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h-1}{h+1}+1}{h} = \lim_{h \to 0} \frac{2}{h+1} = 2 \quad //$$

$$\boxed{\frac{\frac{h-1}{h+1}+1}{h} = \frac{1}{h}\left(\frac{h-1}{h+1}+1\right) = \frac{1}{h}\left(\frac{h-\cancel{1}+h+\cancel{1}}{h+1}\right) = \frac{1}{\cancel{h}}\cdot\frac{2\cancel{h}}{h+1} = \frac{2}{h+1}}$$

eqn of tangent line:

$(0,-1)$ slope $= 2$

$y+1 = 2(x-0)$

$y = 2x-1$ //

$Q(1.0) = 1.09\cdots$

point $(3, g(3)) = (3, \sqrt{3})$

secant line $y = \underbrace{\sqrt{3}}_{g(3)} + Q(1.)(x-3)$

Summary:

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

1. Slope of tangent line to $f(x)$ at $x = x_0$

2. Equation of tangent line is

$$y = f(x_0) + (\underline{\text{slope}})(x - x_0)$$

3. The slope of tangent line is the instantaneous rate of change of f with respect to x <u>at</u> $x = x_0$