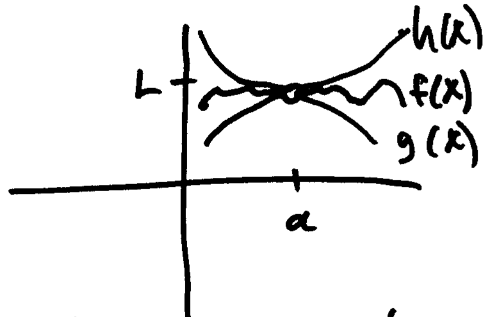
From last time:

Calculating limits.

1. Direct substitution
2. Algebraic simplification $\frac{0}{0}$

$\lim_{x \to a} \frac{P(x)}{Q(x)} \to \frac{0}{0}$ then $P(x)$ & $Q(x)$ have a factor of $x-a$.

3. Sandwich Thm.




$\lim_{x \to a} f(x) = L$

Know: $\lim_{x \to a} h(x) = L = \lim_{x \to a} g(x)$ <u>and</u>

$g(x) \le f(x) \le h(x)$

Then: $\lim_{x \to a} f(x) = L$

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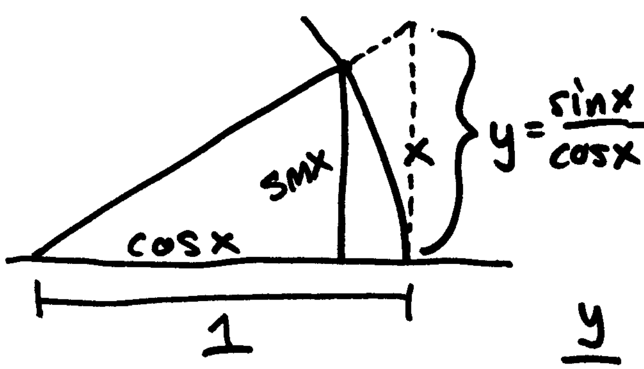
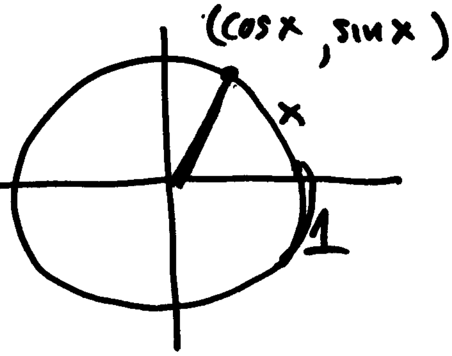
$\lim_{x \to 0} \frac{\sin(x)}{x}$ use trigonometry:




$\frac{y}{\sin x} = \frac{1}{\cos x}$ $\quad y = \frac{\sin x}{\cos x}$

area of small $\Delta$ $\leq$ area of sector $\leq$ area of big $\Delta$

$(\pi)\left(\begin{array}{c}\text{fraction of} \\ \text{circle corresp to } x\end{array}\right)$

$(\pi)\left(\frac{x}{2\pi}\right)$

$$\underbrace{\cancel{\tfrac{1}{2}}\cos x \cdot \sin x \leq \frac{x}{\cancel{2}}} \qquad \frac{x}{\cancel{2}} \leq \cancel{\tfrac{1}{2}}\frac{\sin x}{\cos x}$$

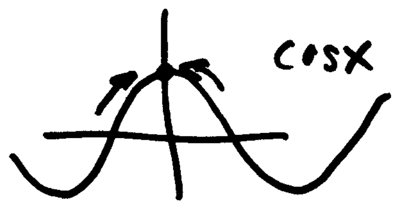
$$\frac{\sin x}{x} \leq \frac{1}{\cos x} \qquad \cos x \leq \frac{\sin x}{x}$$

$$\boxed{\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}}$$

$$\lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to 0} \frac{1}{\cos x} = 1$$

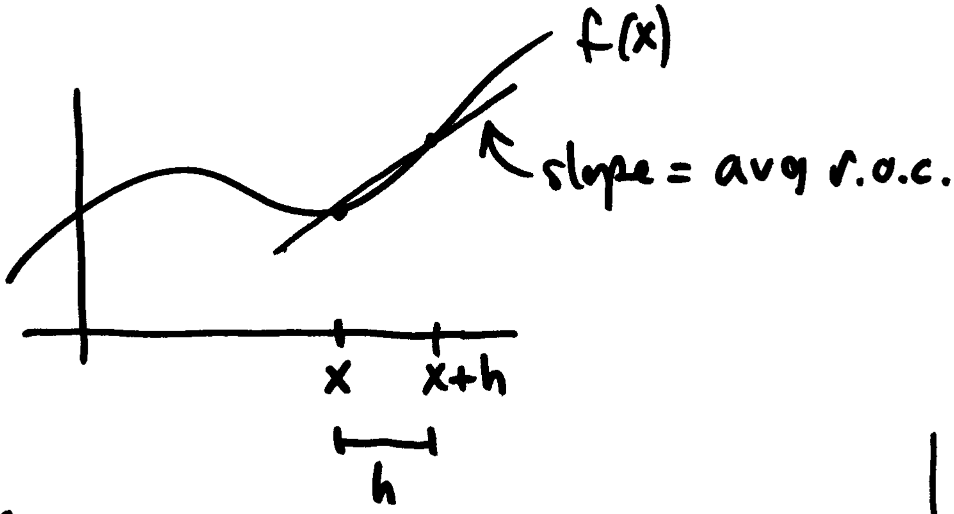
$\boxed{\begin{array}{l}\text{Thm 1} \\ \text{p 78} \\ \text{\#5}\end{array}}$




$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

eg #46) $f(x) = \frac{1}{x}$ $\quad x = -2$

Want $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ = inst. r.o.c. of f <u>at</u> x.

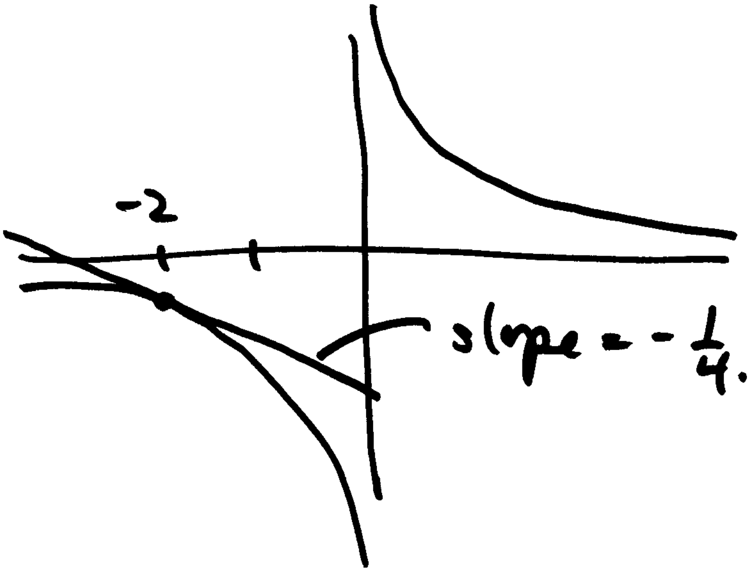



Avg r.o.c. of f on $[x, x+h]$ $= \frac{f(x+h) - f(x)}{h}$

$$\frac{f(x+h) - f(x)}{h} = \frac{f(-2+h) - f(-2)}{h} = \frac{\frac{1}{-2+h} - \frac{1}{-2}}{h}$$

$$= \frac{1}{h}\left(\frac{1}{h-2} + \frac{1}{2}\right) = \frac{1}{\cancel{h}}\left(\frac{(\cancel{2} + \cancel{h} - \cancel{2})}{2(h-2)}\right) = \frac{1}{2(h-2)}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{2(h-2)} = -\frac{1}{4}$$




$f(x) = \frac{1}{x}$

#4(e) $f(x) = \sqrt{3x+1} \qquad x = 0$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3h+1}-1}{h} = \lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{2}$$

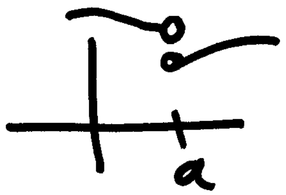
$$\frac{\sqrt{3h+1}-1}{h} \cdot \frac{\sqrt{3h+1}+1}{\sqrt{3h+1}+1} = \frac{3\cancel{h}+\cancel{1}-\cancel{1}}{\cancel{h}(\sqrt{3h+1}+1)}$$

$$= \frac{3}{\sqrt{3h+1}+1}$$

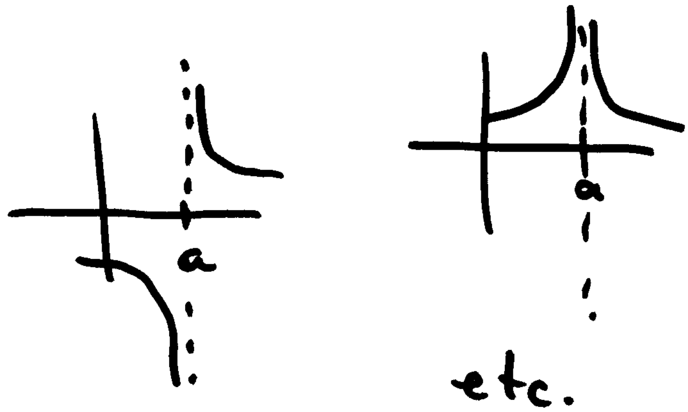
## 2.4 One-sided limits + limits at $\infty$.

### 1. One-sided limits

How can a limit fail to exist?

a. 


one sided limits differ

b. 

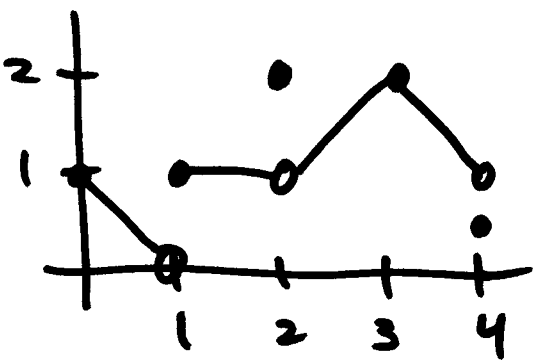

infinite limits

etc.

c. 


oscillation

e.g. ex 2 p 97




$\lim_{x \to 1} f(x) = DNE$

$\lim_{x \to 2} f(x) = 1$

$\lim_{x \to 3} f(x) = 2$

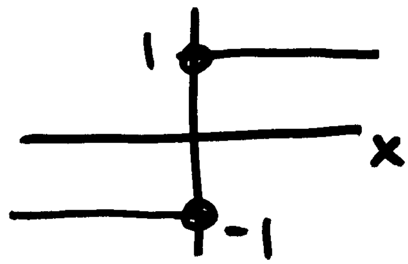
$\lim_{x \to 4} f(x) = DNE$

Can describe further:

$$\lim_{x \to 1^-} f(x) = 0 \qquad \lim_{x \to 1^+} f(x) = 1$$

$$\lim_{x \to 4^-} f(x) = 1$$

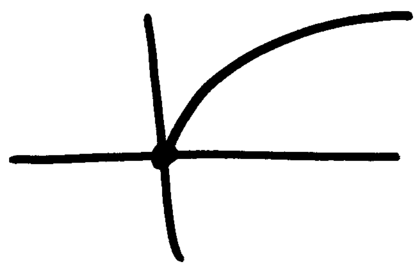
<u>e.g.</u> $\lim_{x \to 0} \frac{x}{|x|}$ DNE




$$\lim_{x \to 0^+} \frac{x}{|x|} = 1 \qquad \lim_{x \to 0^-} \frac{x}{|x|} = -1$$

<u>e.g.</u> $\lim_{x \to 0} \sqrt{x}$

DNE



$$\lim_{x \to 0^+} \sqrt{x} = 0$$

#24)

Know: $\lim\limits_{\theta \to 0^{\pm}} \frac{\sin\theta}{\theta} = 1$

$$\lim_{h \to 0^-} \frac{3h}{\sin(3h)} \cdot \frac{1}{3} = \frac{1}{3} \cdot \underbrace{\lim_{h \to 0^-} \frac{3h}{\sin(3h)}}_{1} = \frac{1}{3}$$

$$\lim_{h \to 0^-} \frac{\sin(3h)}{3h} = 1$$

$$\lim_{h \to 0^-} \frac{3h}{\sin(3h)} = 1$$

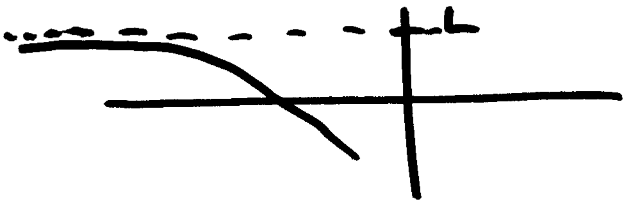
$\lim\limits_{x \to a} f(x) = L \qquad \lim\limits_{x \to a} \frac{1}{f(x)} = \frac{1}{L}$

2. Limits at $\infty$.

$\lim_{x\to\infty} f(x)$ $\qquad$ $\lim_{x\to-\infty} f(x)$

$\lim_{x\to\infty} f(x) = L$


$\lim_{x\to-\infty} f(x) = L$




We say that $f(x)$ has a <u>horizontal asymptote</u> at the line $y=L$.

Basic example:

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

| $x$ | $\frac{1}{x}$ |
|---|---|
| 1 | 1 |
| 10 | .1 |
| 100 | .01 |
| 5000 | .0002 |
| 1000000 | .000001 |

Idea: If denominator gets big + numerator is constant then fraction = $\frac{\text{numerator}}{\text{denominator}}$ gets small.

eg $\lim_{x \to \infty} \frac{3}{x^{2/3}} = 0$

e.g. $\lim_{x \to \infty} \frac{\sin(x)}{x} = 0$

Sandwich: $-1 \le \sin(x) \le 1$

$$\frac{-1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}$$

e.g #38 p 108

$\lim_{x \to \infty} \left(\pi - \frac{2}{x^2}\right) = \pi \qquad \lim_{x \to -\infty} \left(\pi - \frac{2}{x^2}\right) = \pi$

#40 $\lim_{x \to \infty} \frac{1}{8-\left(\frac{5}{x^2}\right)} = \frac{1}{8}$

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eg #52

$$\lim_{x \to \infty} \frac{2x^3+7}{x^3-x^2+x+7} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{2+(7/x^3)}{1-(1/x)+(1/x^2)+(7/x^3)} = 2$$

---

$$\lim_{x \to \infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x \to \infty} \frac{2\cancel{x^3}}{\cancel{x^3}} = 2$$

$x = 100$

$x^3 = 1000000$

$x^2 = 10000$

$x = 100$

$7 = 7$

#54)

$$\lim_{x\to\infty} \frac{3x+7}{x^2-2} = \lim_{x\to\infty} \frac{3x}{x^2}$$

$$= \lim_{x\to\infty} \frac{3}{x} = 0$$

$$\lim_{x\to-\infty} \frac{3x+7}{x^2-2} = \lim_{x\to-\infty} \frac{3x}{x^2} = \lim_{x\to-\infty} \frac{3}{x} = 0$$

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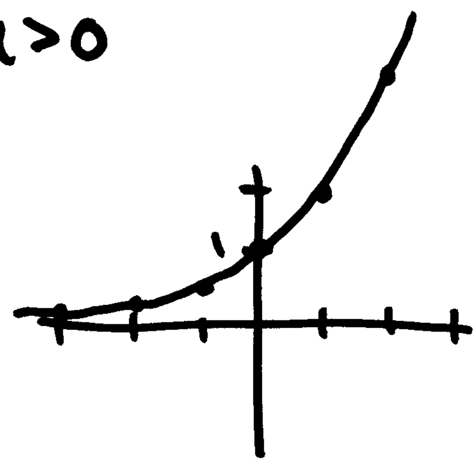
## EXPONENTIAL FUNCTIONS 1.5

$f(x) = a^x \quad a > 0$

$f(x) = 2^x$



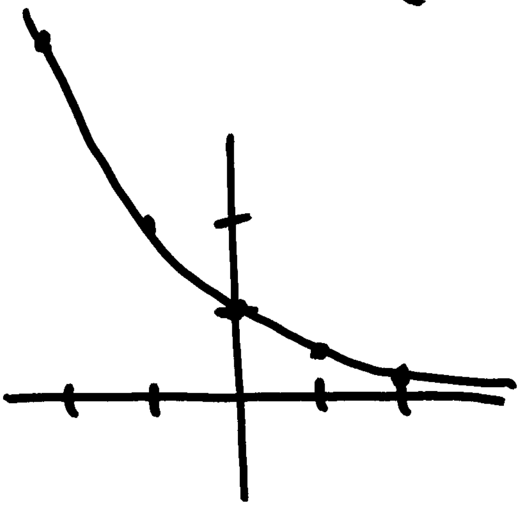

| $x$ | $2^x$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| −1 | $2^{-1} = \frac{1}{2}$ |
| 2 | 4 |
| −2 | $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ |
| 3 | 8 |
| −3 | $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ |

$$\lim_{x\to-\infty} 2^x = 0$$

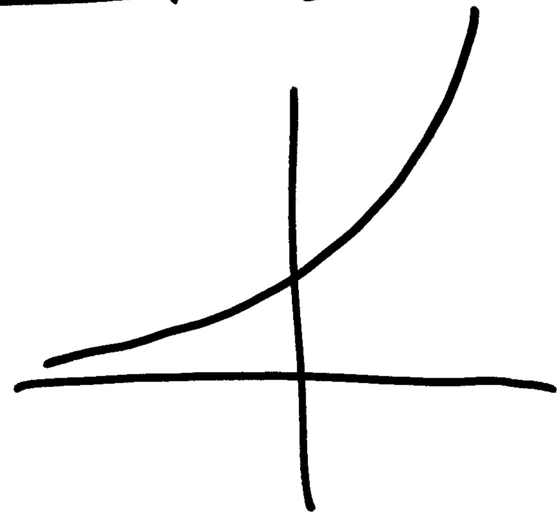
$$\lim_{x\to\infty} 2^x \text{ DNE}$$

(is infinite)

$$f(x) = 2^{-x} = (2^{-1})^x = \left(\tfrac{1}{2}\right)^x = \frac{1}{2^x}$$



$$2^{-x} = \left(\tfrac{1}{2}\right)^x$$



$$2^x$$

| $x$ | $\left(\frac{1}{2}\right)^x$ |
| --- | --- |
| 0 | 1 |
| 1 | $\frac{1}{2}$ |
| $-1$ | $\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$ |
| 2 | $\frac{1}{4}$ |
| $-2$ | 4 |

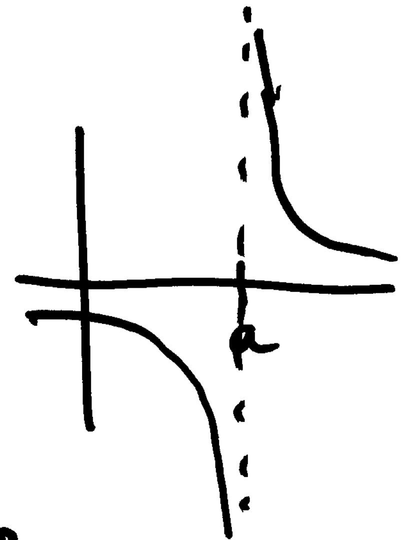
$$\lim_{x \to \infty} a^x = 0 \quad \text{if} \quad 0 < a < 1$$

$$\lim_{x \to -\infty} a^x = 0 \quad \text{if} \quad a > 1$$

$$f(x) = e^x \qquad e \approx 2.71828$$

## 2.5 Infinite limits + vertical asymptotes

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$




$$\lim_{x \to a^{+}} f(x) = \infty$$

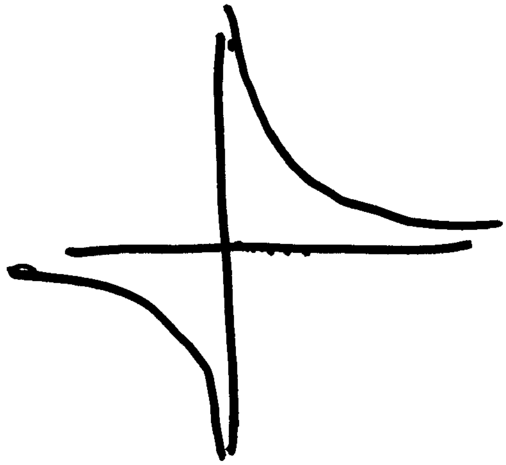
$$\lim_{x \to a^{-}} f(x) = -\infty$$

We say that as $x \to a$ $f(x)$ increases ($+\infty$) or decreases ($-\infty$) without bound.

If $\lim_{x \to a^{\pm}} f(x) = \pm\infty$ we say that graph of $f$ has a vertical asymptote at $x = a$.

Basic example:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$



$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

| $x$ | $\frac{1}{x}$ |
| --- | --- |
| -1 | -1 |
| -.5 | -2 |
| -.1 | -10 |
| -.05 | -20 |
| -.01 | -100 |
| -.001 | -1000 |

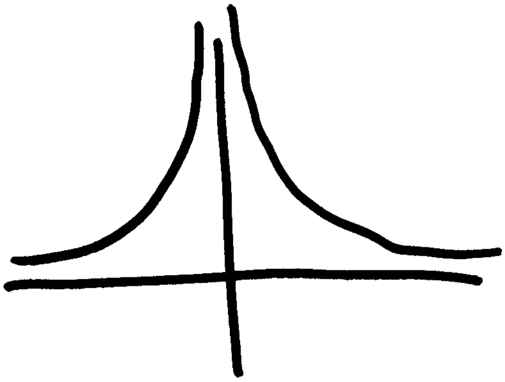
If denominator becomes <u>small</u> (i.e. close to zero) and numerator is constant then fraction $= \frac{\text{numerator}}{\text{denominator}}$ becomes <u>infinite</u>.

In previous case:

$$\lim_{x \to 0} \frac{1}{x} \quad \text{DOES NOT EXIST}$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

<u>e.g.</u> $\lim_{x \to 0} \frac{1}{x^2} = \infty$



$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty$$

| $x$ | $\frac{1}{x^2}$ |
|---|---|
| $-1$ | $1$ |
| $-0.1$ | $100$ |
| $-0.05$ | $400$ |
| $-.001$ | $10000$ |

$$\lim_{x \to 0^-} \frac{1}{x^2} = \infty$$

eg #6 p 117

$$\lim_{x\to -5^-} \frac{3x}{2x+10} = \infty$$

- - - - - - - - -

-7 -6 -5 -4 $\quad 3x$

- - - - - - - 0 + + +

-7 -6 -5 $\quad 2x+10$

#8 $$\lim_{x\to 0} \frac{-1}{x^2(x+1)} = -\infty$$

$$\lim_{x\to 0^+} \frac{-1}{x^2(x+1)} = -\infty$$

+ + + 0 + + + + $\quad x^2$

0 + + + + + + $\quad x+1$

-1 0

$$\lim_{x\to 0^-} \frac{-1^-}{\underset{+}{x^2}\underset{+}{(x+1)}} = -\infty$$

- - + + + $\quad x+1$

-1 0

Finding vertical asymptotes:

If $\lim_{x \to a^{\pm}} f(x) = \pm\infty$ then $f(x)$ has a vertical asymptote at $x=a$.

---

What ftns have vertical asymptotes?

(a) Rational functions

$$f(x) = \frac{P(x)}{Q(x)}$$

If at $x=a$, $P(a) \neq 0$ and $Q(a) = 0$ then $f$ has a v.asy. at $x=a$.

eg. $f(x) = \frac{x}{x^2-1}$ Find v. asy.

$x^2-1=0$

$(x+1)(x-1)=0$

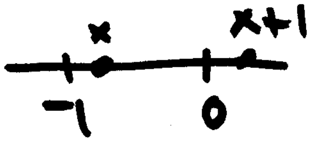
$x=1$ $x=-1$

vert. asymp.

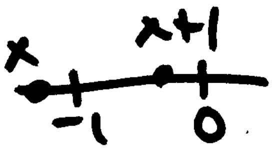
$$\lim_{x \to 1^+} \frac{x}{x^2-1} = \lim_{x \to 1^+} \frac{x}{(x+1)(x-1)} = \infty$$

$$\lim_{x \to 1^-} \frac{x}{(x+1)(x-1)} = -\infty$$

$$\lim_{x\to -1^+} \frac{x^{(-)}}{\underset{(+)}{(x+1)}\underset{(-)}{(x-1)}} = \infty$$

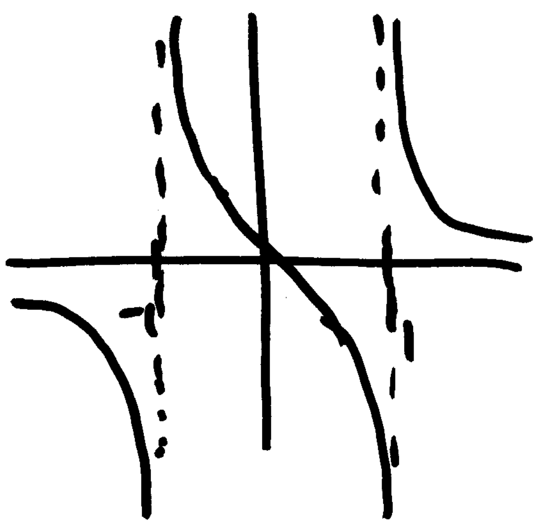



$$\lim_{x\to -1^-} \frac{x^{(-)}}{\underset{(-)}{(x+1)}\underset{(-)}{(x-1)}} = -\infty$$




$x = -1.1$

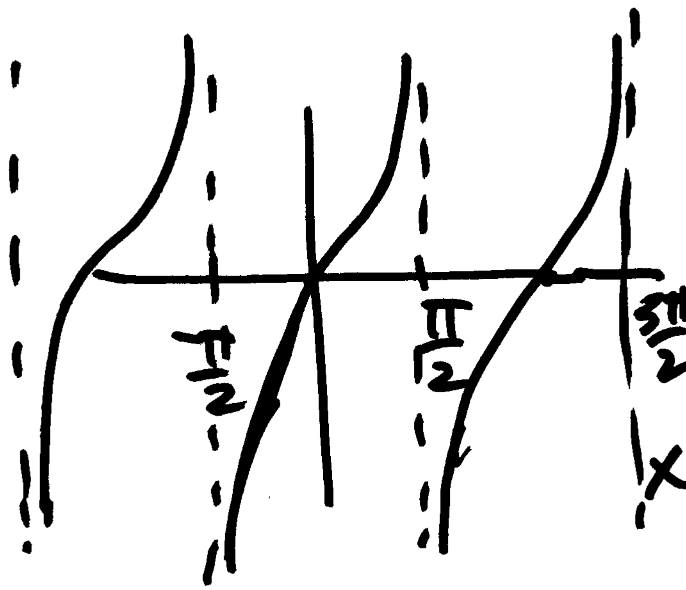
$x+1 = -.1$

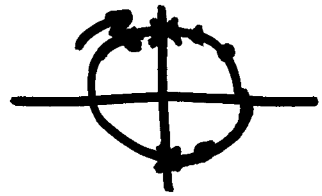


(b) some trig functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$\cos(x) = 0$ when $x = \frac{\pi}{2}, -\frac{\pi}{2}$



$$\lim_{x \to \frac{\pi}{2}^-} \tan(x)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sin(x)\ (+)}{\cos(x)\ (+)}$$

$$= \infty$$

$$\lim_{x \to -\frac{\pi}{2}^+} \frac{\sin(x)^{(-)}}{\cos(x)_{(+)}} = -\infty$$

$$\lim_{x \to \frac{\pi}{2}^+} \frac{\sin(x)^{(+)}}{\cos(x)_{(-)}} = -\infty$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$