MATH 113-A01

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2.1 Rates of Change + Limits

Rates of change

$$y = f(x)$$

want the rate of change of y with respect to x.

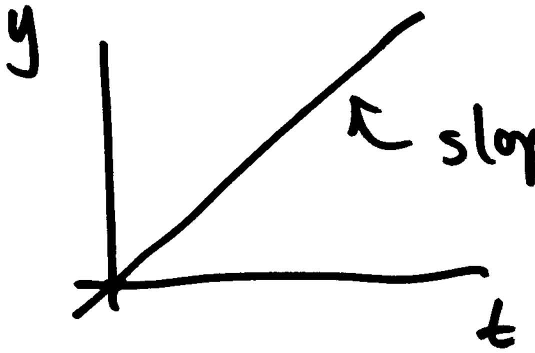
1. Easy example:

y = reading on a car's trip odometer (in miles)

t = elapsed time from start of trip (in hours)

Spse car moves at const. speed, say 30 miles per hour.

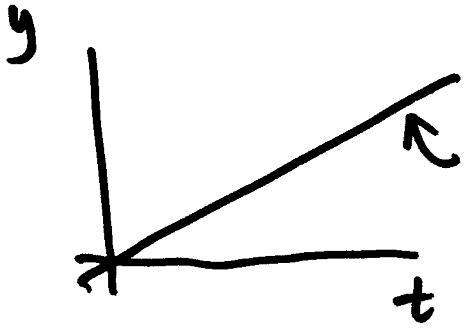
$$y = 30t$$



Say: the rate of change of y w.r.t. t is $30 \; \dfrac{\text{miles}}{\text{hour}}$ ← units of y, ← units of t

2. Measuring rate of change if unknown.

Assume constant r.o.c.: $y = mt$ ← m: unknown r.o.c.



a. choose 2 distinct times t_1, t_2

b. record odometer at t_1 and t_2

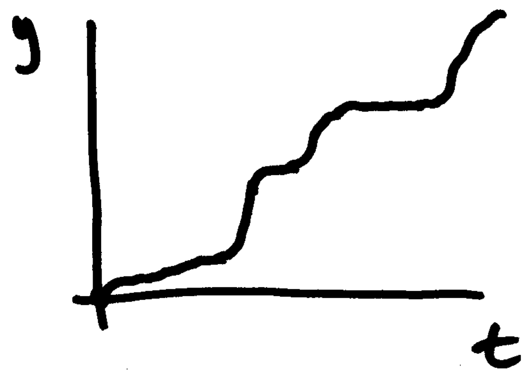
$$y_1 = mt_1 \qquad y_2 = mt_2$$

c. Find average rate of change:

$$\frac{y_2 - y_1}{t_2 - t_1} = \frac{mt_2 - mt_1}{t_2 - t_1} = \frac{m(t_2 - t_1)}{(t_2 - t_1)} = m$$

d. since r.o.c. is constant then this is enough.
(r.o.c. constant $\Longleftrightarrow$ slope is constant)

3. What if r.o.c. is <u>not</u> constant?



$y = f(t)$

Need: A notion of <u>slope</u> and <u>rate of change</u> when function is <u>not linear</u>.

<u>e.g.</u> Free fall.

y = distance dropped object has fallen. (in feet)

t = elapsed time since object dropped (in seconds).

$$y = 16t^2 \quad \text{(not linear)}$$

Q: How fast is object moving at $t=2$ seconds?
(units: feet/second)

How to do this?

a. pick 2 distinct times t_1 and t_2

Lets take $t_1 = 2$ $t_2 = t$

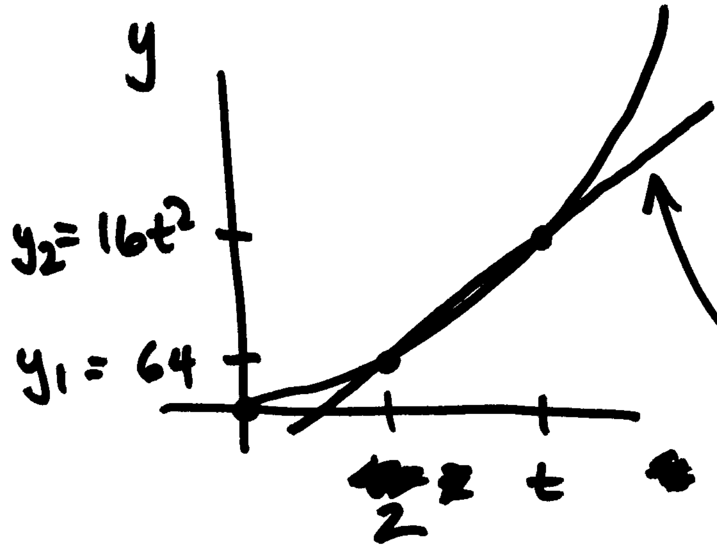
b. $y_1 = f(t_1) = f(2)$
$y_2 = f(t_2) = f(t)$

Average r.o.c.:

$$\frac{y_2 - y_1}{t_2 - t_1} = \frac{f(t) - f(2)}{t-2} = \frac{16t^2 - 64}{t-2}$$

$$= 16 \cdot \frac{t^2 - 4}{t-2}$$

Intuition: If t is close to 2 then average r.o.c. gives better estimate to actual r.o.c. at $t=2$.



$y = 16t^2$

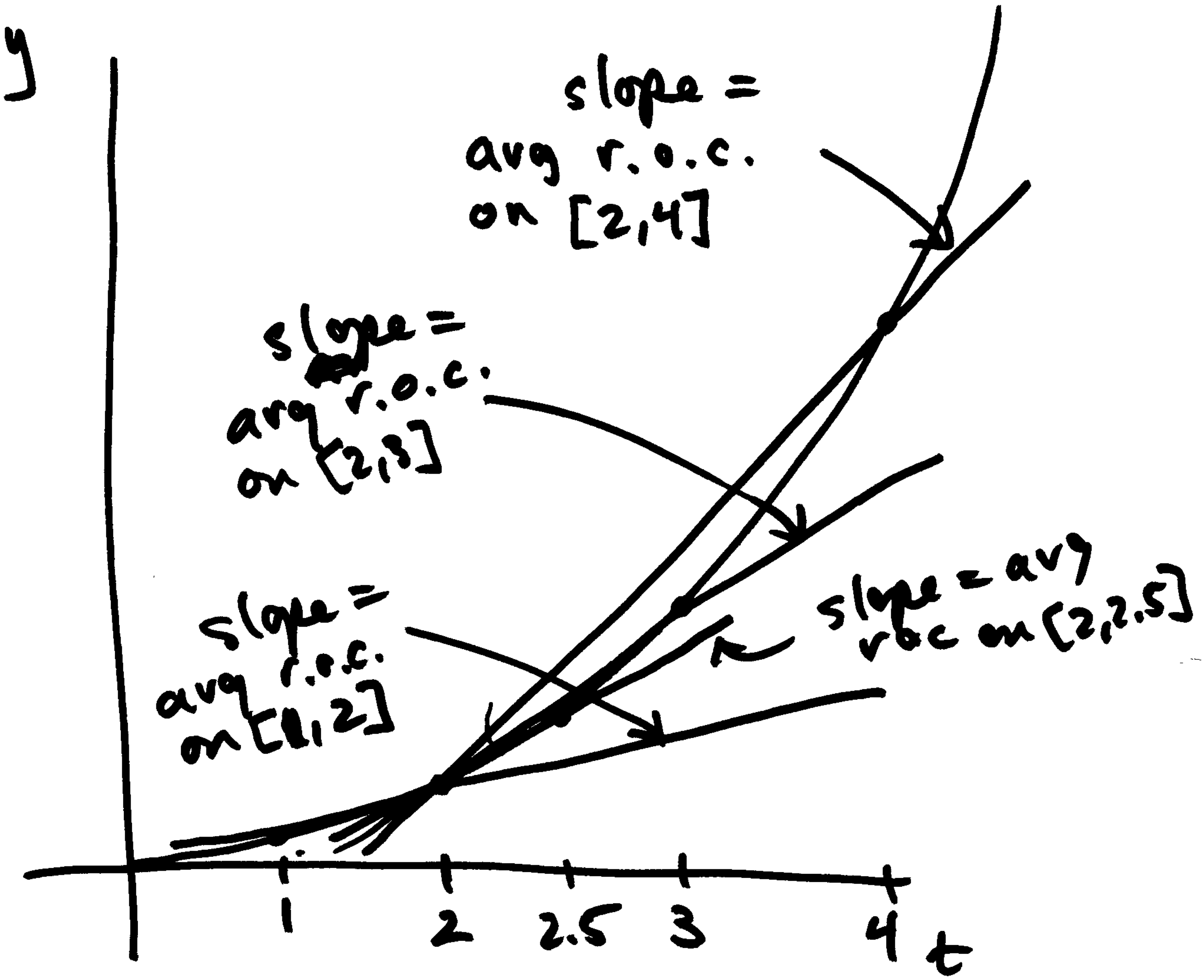
slope = avg. rate of change between $t_1 = 2$, $t_2 = t$.

Avg r.o.c. between $t_1 = 2$ $t_2 = t$

$$16 \cdot \left(\frac{t^2 - 4}{t - 2} \right)$$

t	$\left(\frac{t^2-4}{t-2}\right)$ $\frac{\text{feet}}{\text{sec.}}$
3	$\frac{9-4}{3-2} = 5$
2.5	$\frac{6.25-4}{2.5-2} = \frac{2.25}{.5} = 4.5$
1.5	3.5
1	$\frac{1-4}{1-2} = 3$
2.1	4.1
1.9	3.9
2.05	4.05
1.95	3.95

Graphically:



These lines are <u>secant</u> lines

LIMIT

we are trying to evaluate

$$f(t) = \frac{t^2-4}{t-2}$$

at $t=2$.

$$f(2) = \frac{0}{0} \quad \text{not defined.}$$

Limits sometimes allow such quantities to be evaluated.

Idea: Evaluate $f(t)$ for t near 2 but not on 2.

t	$f(t)$
3	5
2.5	4.5
2.1	4.1
2.05	4.05
2.01	4.01
2.005	4.005

We cant evaluate $f(2)$

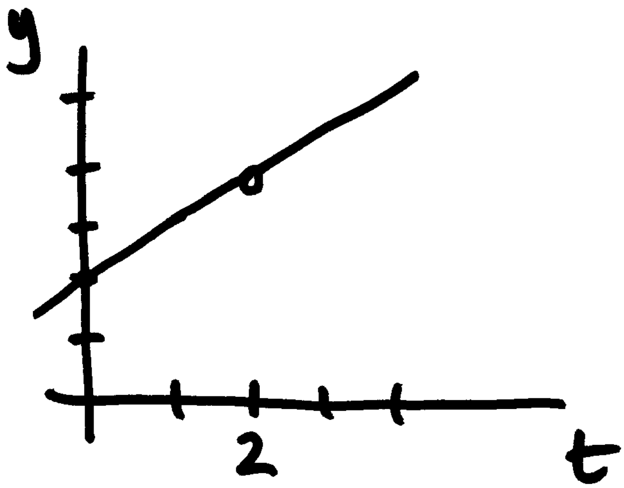
but looks like $f(2)$ wants to be 4.

What is happening?

Algebraic simplification:

$$\frac{t^2-4}{t-2} = \frac{(t+2)(t-2)}{(t-2)} = t+2$$

when $t \neq 2$



$$y = \frac{t^2-4}{t-2}$$

We say

$$\boxed{\lim_{t \to 2} \frac{t^2-4}{t-2} = 4}.$$

Free fall: $y = 16t^2$

instantaneous rate of change of y w.r.t. t at $t=2$ is

$16 \cdot 4 = 64$ feet/sec.

Basic properties of limits

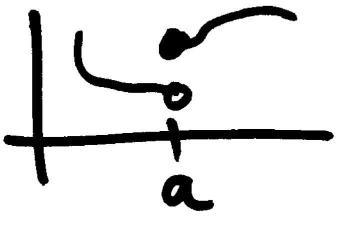
$$\lim_{x \to a} f(x) = L$$

1. Value of $\lim_{x \to a} f(x)$ does not depend on value of $f(x)$ <u>at</u> $x=a$ but only <u>near</u> $x=a$.

2. Limits need not exist. How can it fail?

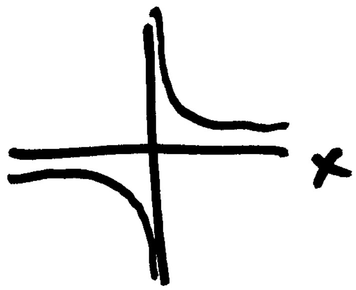
eg #2 p.75

a. one-sided limits can differ



$\lim_{x \to a} f(x)$ does not exist.

b. limit can be infinite



e.g. $f(x) = \frac{1}{x}$

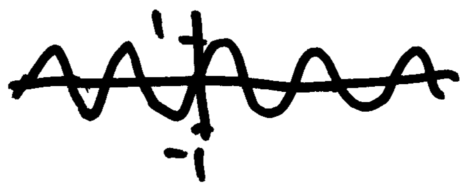
$\lim_{x \to 0} f(x)$ does not exist

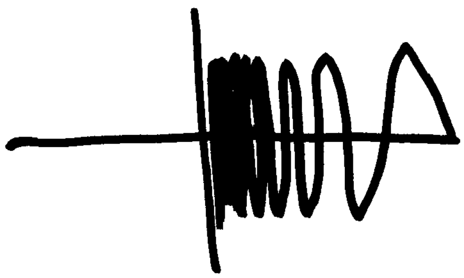
c. function can oscillate

$\sin\left(\frac{1}{x}\right)$

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

sin(t)





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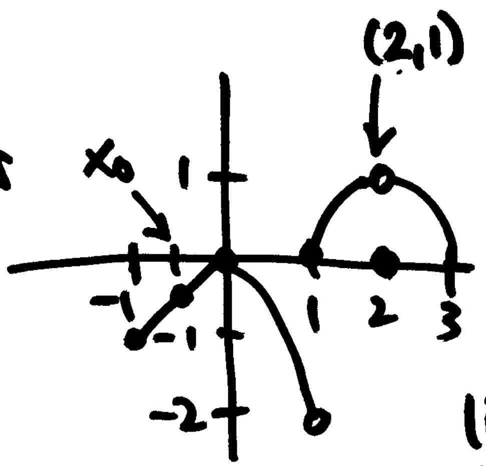
eg #4

$\lim_{x \to x_0} f(x)$ exists for x_0 in $(-1,1)$ ✓



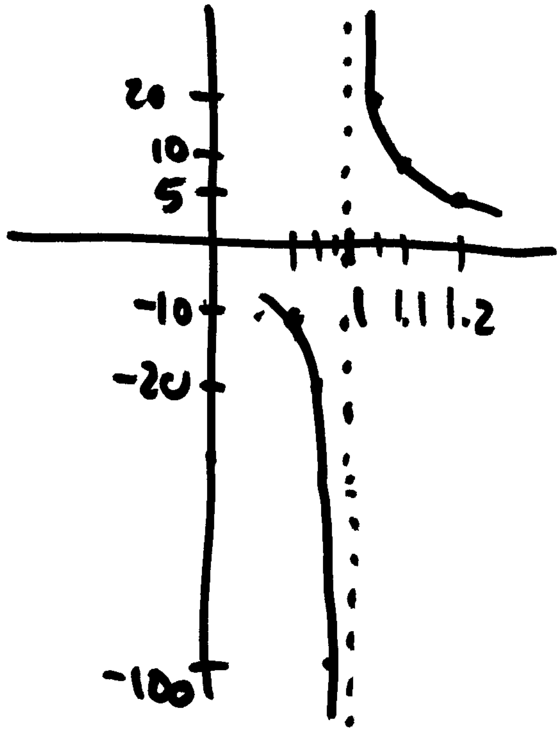
$\lim_{x \to 2} f(x) = 1$ ✓

$\lim_{x \to 2} f(x) = f(2)$ ✗

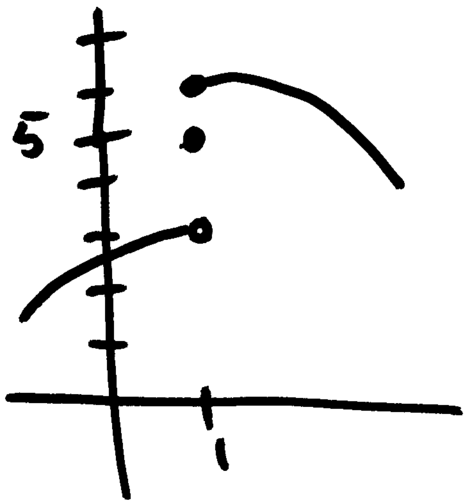
$\lim_{x \to 1} f(x)$ DOES NOT EXIST ✓

eg $\lim_{x \to 1} \dfrac{1}{x-1}$ DOES NOT EXIST

x	$\frac{1}{x-1}$
1.2	5
1.1	10
1.05	20
.9	−10
.95	−20
.99	−100

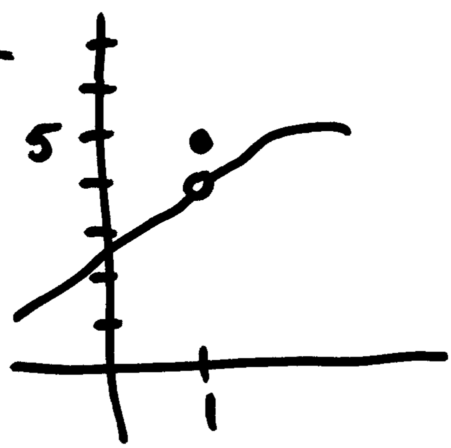


#10



$f(1) = 5$

$\lim_{x \to 1} f(x)$ does not exist.



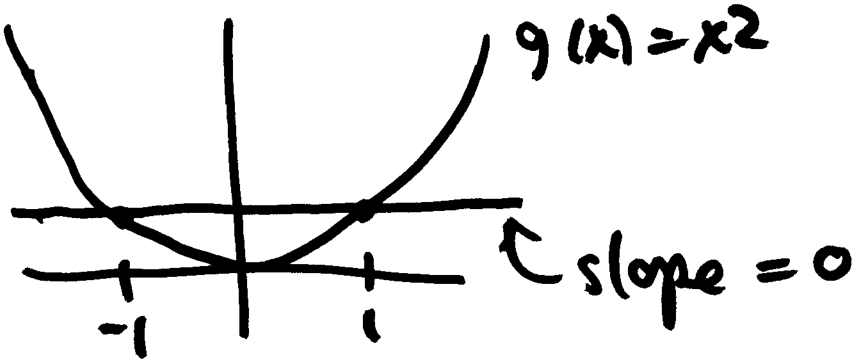
$\lim_{x \to 1} f(x) = 4$

$f(1) = 5$

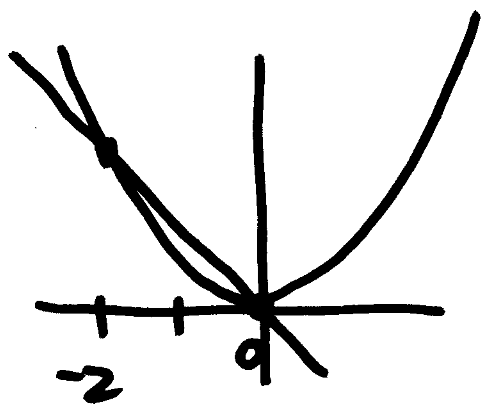
eg #30 $g(x) = x^2$

$[-1,1]$ Avg rate of change:

$$\frac{g(1)-g(-1)}{1-(-1)} = \frac{1-1}{2} = \frac{0}{2} = 0$$



$[-2,0]$



$$\frac{g(0)-g(-2)}{0-(-2)}$$

$$= \frac{0-4}{2} = -2$$

$$\frac{g(-2)-g(0)}{-2-0} = \frac{4-0}{-2}$$

$$= -2$$

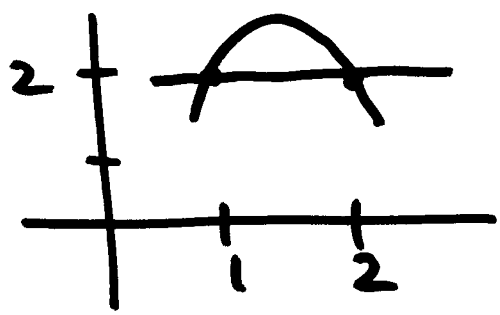
eg #34 $P(\theta)=\theta^3-4\theta^2+5\theta$

$[1,2]$

$$\text{Avg r.o.c.} = \frac{P(2)-P(1)}{2-1}$$

$$= \frac{(2^3-4\cdot 2^2+5\cdot 2)-(1^3-4\cdot 1^2+5\cdot 1)}{2-1}$$

$$= \frac{2-2}{2-1} = \frac{0}{1} = 0$$



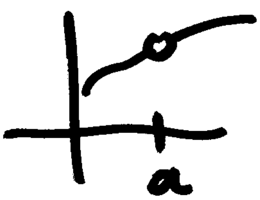
2.2 Limit Laws

How to calculate limits.

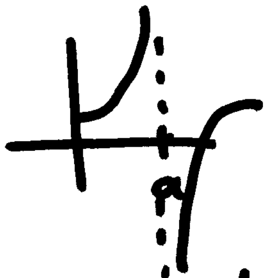
$$\lim_{x \to a} f(x)$$

1. Sometimes $\lim_{x \to a} f(x) = f(a)$. When?

If $f(x)$ has no <u>holes</u> or <u>asymptotes</u> at $x = a$ then $\lim_{x \to a} f(x) = f(a)$



hole



asymptote

specifically (for example)

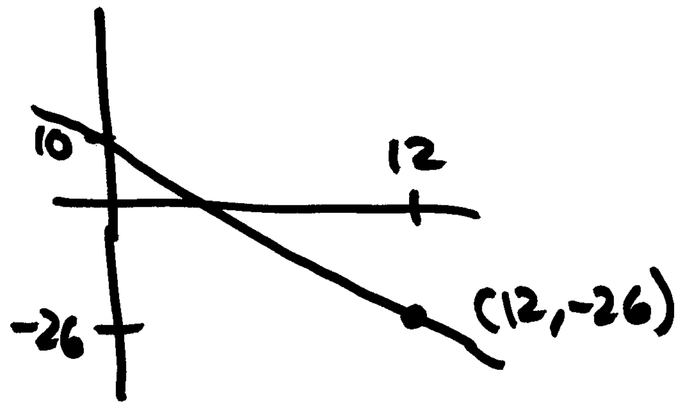
a) If $f(x)$ is a polynomial

b) If $f(x) = \frac{P(x)}{Q(x)}$, P, Q polynomials

<u>and</u> $\lim_{x \to a} Q(x) = Q(a) \neq 0$

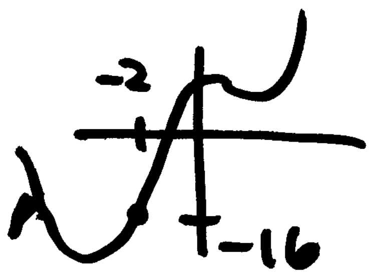
c) If $f(x)$ is a rational expression involving ~~~~ powers of polynomials (I mean fractional powers) as long as these are defined.

e.g. #2 $\lim_{x \to 12} (10-3x) = -26 = (10-3(12))$



#4. $\lim_{x \to -2} x^3 - 2x^2 + 4x + 8$

$= -8 - 2(4) + 4(-2) + 8 = -16$



#8 $\lim_{x \to 5} \frac{4}{x-7} = \frac{4}{-2} = -2$

#14 $\lim_{z \to 0} (2z-8)^{1/3} = (-8)^{1/3} = -2$

#16 $\lim\limits_{h\to 0} \dfrac{5}{\sqrt{5h+4}+2} = \dfrac{5}{4}$

2. Algebraic simplification

<u>eg</u> #20

$$\lim_{x\to -3} \frac{x+3}{x^2+4x+3} \qquad \left(\frac{0}{9-12+3} = \frac{0}{0}\right)$$

$$\left[\ \frac{x+3}{x^2+4x+3} = \frac{\cancel{x+3}}{(x+1)\cancel{(x+3)}} = \frac{1}{x+1} \quad \text{if } x \neq -3\ \right]$$

$$= \lim_{x\to -3} \frac{1}{x+1} = -\frac{1}{2}$$

P, Q polynomials

$$\boxed{\text{If } \lim_{x\to a} \frac{P(x)}{Q(x)} \text{ evaluates to } \frac{0}{0} \text{ then } P, Q \text{ have a common factor of } (x-a)}$$

#28 $\lim\limits_{v \to 2} \dfrac{v^3-8}{v^4-16}$

$$\frac{v^3-8}{v^4-16} = \frac{(v-2)(v^2+2v+4)}{(v^2-4)(v^2+4)}$$

$$= \frac{\cancel{(v-2)}(v^2+2v+4)}{\cancel{(v-2)}(v+2)(v^2+4)}$$

$$= \frac{v^2+2v+4}{(v+2)(v^2+4)} \quad \text{if } v \neq 2$$

$$\begin{array}{r}
v^2 + 2v + 4 \\
v-2 \overline{\big)\, v^3 + 0v^2 + 0v - 8} \\
-\underline{\;v^3 - 2v^2\qquad\qquad} \\
2v^2 + 0v\qquad \\
-\underline{\;2v^2 - 4v\qquad} \\
4v - 8 \\
\underline{4v - 8} \\
0
\end{array}$$

$$= \lim_{v \to 2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{12}{32} = \frac{3}{8}$$

#18 $\displaystyle\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$

$$\frac{\sqrt{5h+4}-2}{h}\cdot\frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2} = \frac{5\cancel{h}+\cancel{4}-\cancel{4}}{\cancel{h}\left(\sqrt{5h+4}+2\right)}$$

$$\boxed{(x^2-a^2)=(x-a)(x+a)} \qquad = \frac{5}{\sqrt{5h+4}+2}$$

if $h \neq 0$

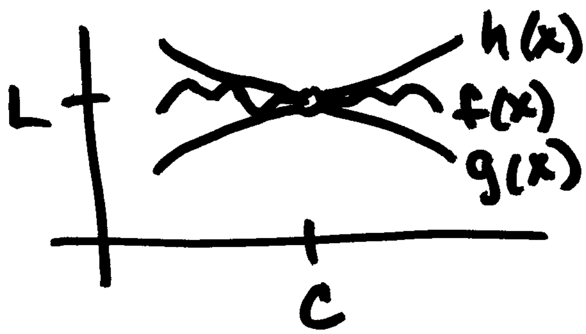
$$= \lim_{h\to 0}\frac{5}{\sqrt{5h+4}+2} = \frac{5}{4}.$$

3. Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then $\lim_{x \to c} f(x) = L$.

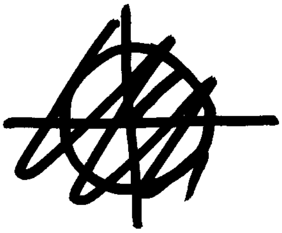


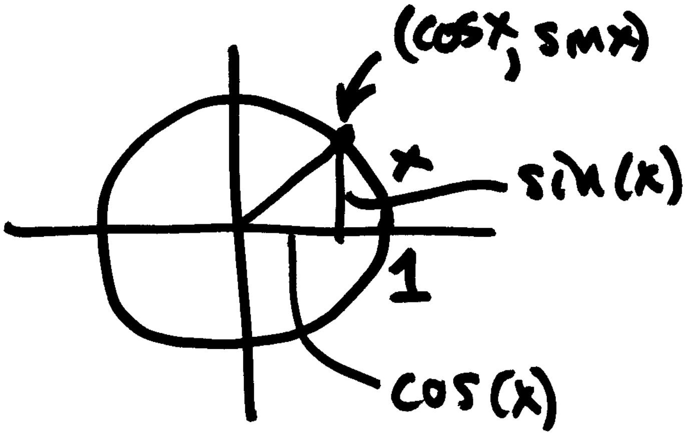
e.g. $\lim_{x \to 0} \frac{\sin(x)}{x}$ $\qquad \frac{0}{0}$

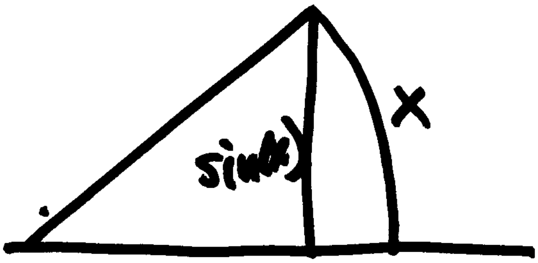
Use Sandwich Thm.

① show $\frac{\sin(x)}{x} \le 1$

or $\sin(x) \le x$

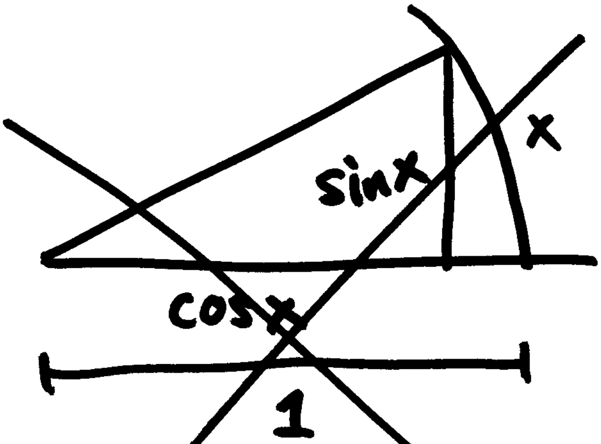




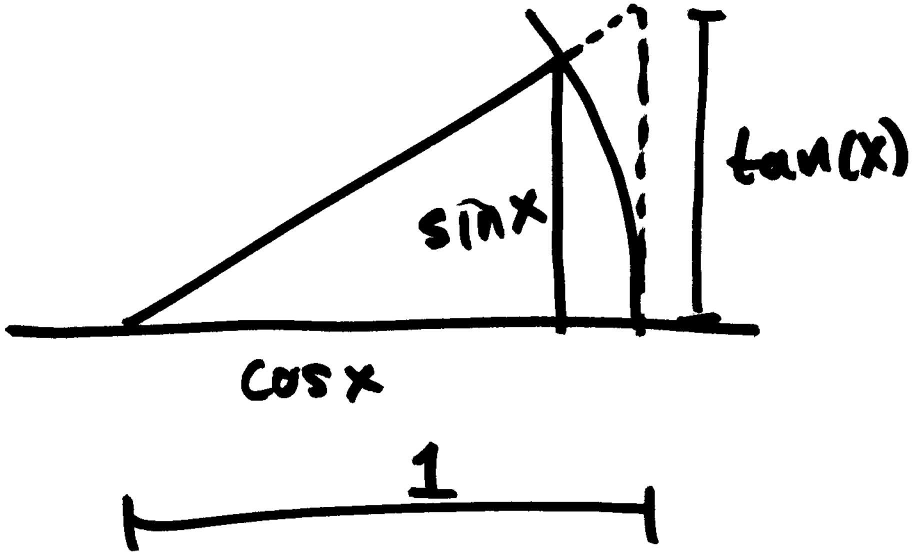


$\therefore$ $\underline{\sin(x) \le x}$

② Show $\cos(x) \le \dfrac{\sin(x)}{x}$



~~area of triangle~~ $\le$ ~~area of segment~~



area of sector $\leq$ area of triangle