FINAL EXAM – TUESDAY JUNE 19

EXAM 4 – 4.5 – 5.5

5.4 36)	4.6 55)
4.5 3)	

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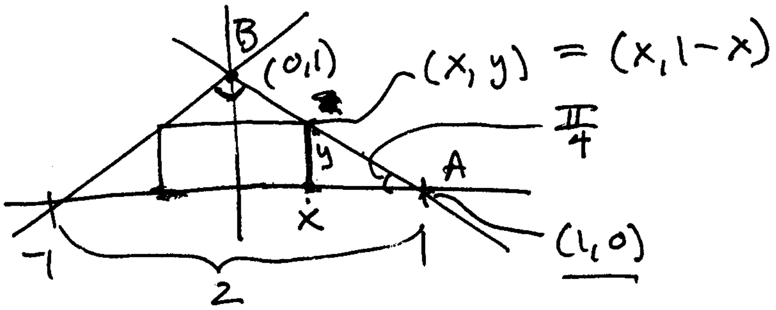
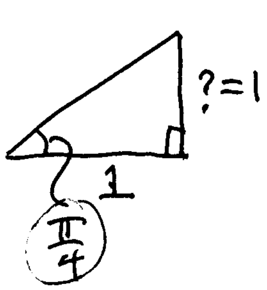


(a) Write y in terms of x.



$\tan\left(\frac{\pi}{4}\right) = \frac{?}{1} = ?$

$\therefore\ ? = 1$

$m = \frac{1-0}{0-1} = -1$ $\qquad$ $y - 0 = -1(x-1)$

$y = 1 - x$

(b) $A = 2x(1-x) = 2x - 2x^2$

(c) Does x have a natural range? $0 \le x \le 1$

Now maximize A on $[0,1]$.

Critical points: $A' = 2 - 4x$ $\qquad$ $2 - 4x = 0$

$x = \frac{1}{2}$

$A(0)=0 \quad A(1)=0 \quad A(\tfrac{1}{2}) = 2(\tfrac{1}{2})(1-\tfrac{1}{2}) = \tfrac{1}{2}$

Largest area: $\frac{1}{2}$

Dimensions: $x=\frac{1}{2} \quad y = 1-\frac{1}{2} = \frac{1}{2}$

so dimensions are $1 \times \frac{1}{2}$

4.6 55) $\lim\limits_{x\to 0^+} x^x$ Form 0^0

$$\boxed{\text{eg. } \lim_{x\to 0^+} x^r = 0 \quad (r>0) \qquad \lim_{x\to 0^+} r^x = 1}$$

$$\lim_{x\to 0^+} x^x = L$$

$$\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x)$$

Form $0\cdot\infty$

$$\lim_{x\to 0^+} \frac{x}{(\ln x)^{-1}} \quad \left(\text{Form } \frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 0^+} \frac{1}{-(\ln x)^{-2}\cdot\frac{1}{x}}$$

$(\ln x)^{-1} = \frac{1}{\ln(x)} \to 0$ as $x\to 0^+$

$$= \lim_{x\to 0} \frac{-x}{(\ln x)^{-2}} \quad \left(\text{Form } \frac{0}{0}\right) \quad \text{NO HELP.}$$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \quad \left(\text{Form } \frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}}$$

$$= \lim_{x\to 0^+} -\frac{1}{x}\cdot x^2 = \lim_{x\to 0^+} (-x) = 0.$$

we have $\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = 0$

$$L = e^{\ln(L)} = e^{\ln\left(\lim_{x\to 0^+} x^x\right)} = e^0 = 1$$

$$\therefore \lim_{x\to 0^+} x^x = 1.$$

5.4 38) $\frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2(y)\,dy\right) = \frac{d}{d\theta}\left(F(\tan\theta) - F(0)\right)$

where $F'(y) = \sec^2(y)$.

$$\int_0^{\tan\theta} \sec^2(u)\,du = \int_0^{\tan\theta} \sec^2(x)\,dx$$

so variable y does not appear.
It is a "placeholder" or "dummy variable"

$$\underline{\sum_{k=1}^{3} k^3} = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = \overline{36}$$

no k here

$$\sum_{j=1}^{3} j^3 = \underline{36} \qquad \sum_{z=1}^{3} z^3 = \underline{36}$$

$$\frac{d}{d\theta}\left(F(\tan\theta) - F(0)\right) = F'(\tan\theta)\cdot\sec^2\theta - 0$$

$$= \sec^2(\tan\theta)\cdot\sec^2(\theta)$$

4.7 8) $\cos(x) = 0$ $\quad x = \frac{\pi}{2}$ is a solution

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ (Solving $f(x)=0$)

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

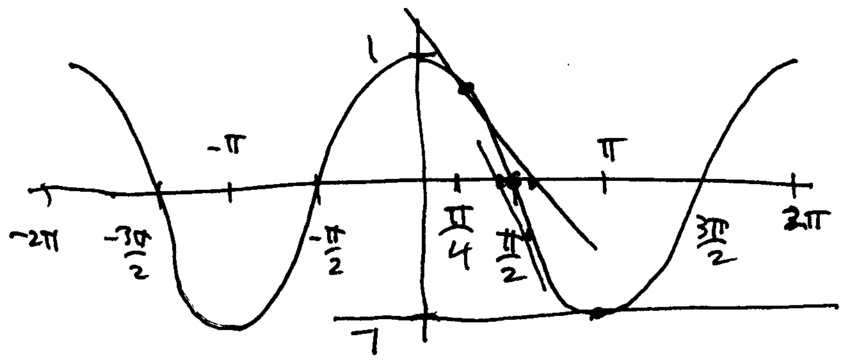
$$x_{n+1} = x_n - \frac{\cos(x_n)}{-\sin(x_n)}$$

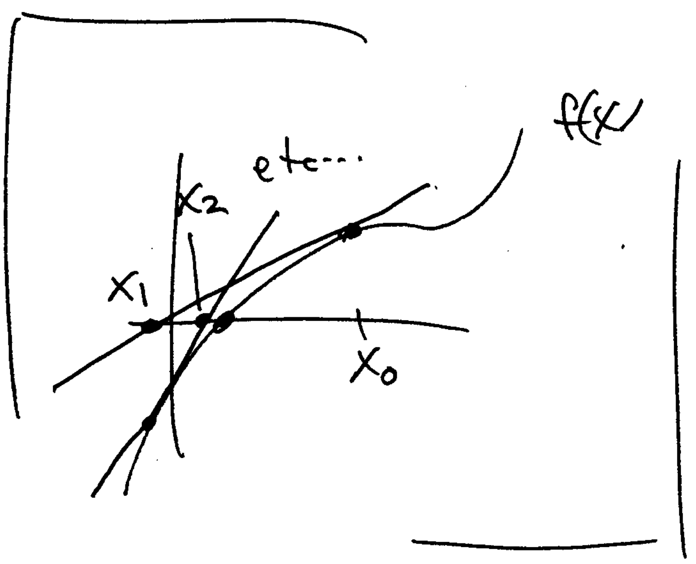
$$= x_n + \cot(x_n)$$

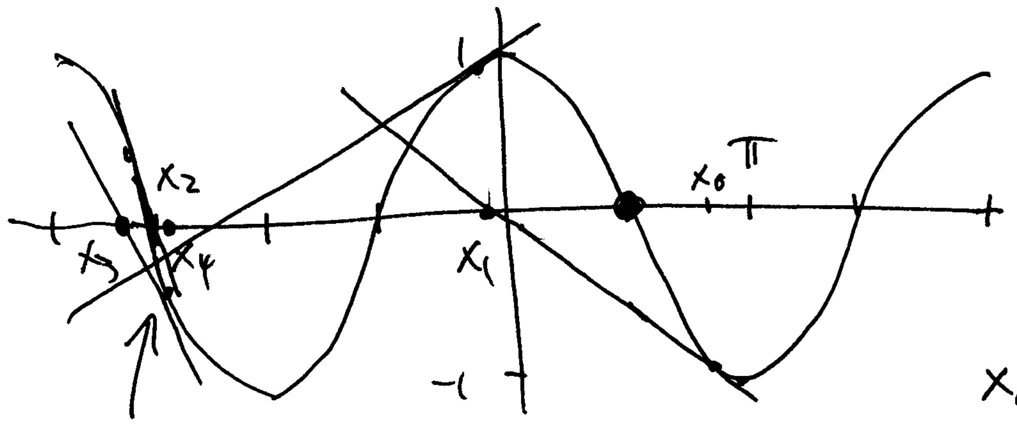
$\underline{x_0 = \pi}$

$x_1 = \pi + \cot(\pi)$

undefined







converging to $-\frac{3\pi}{2}$

$x_0 \neq \pi$
but $x_0 \approx \pi$
and $x_0 < \pi$
say $x_0 = 3.04$

$\int \sin^2(u)\,du$

$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

$= \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2u)\right) du$

$= \int \frac{1}{2}\,du - \frac{1}{2}\int \cos(2u)\,du$

$= \frac{1}{2}u - \frac{1}{2}\cdot\frac{1}{2}\sin(2u) + C$

$= \frac{1}{2}u - \frac{1}{4}\sin(2u) + C$

5.6 Substitution and Area Between Curves

Idea: We have seen how to use substitution for indefinite integrals. How does it work for definite integrals?

① Take a different perspective on substitution.

Recall

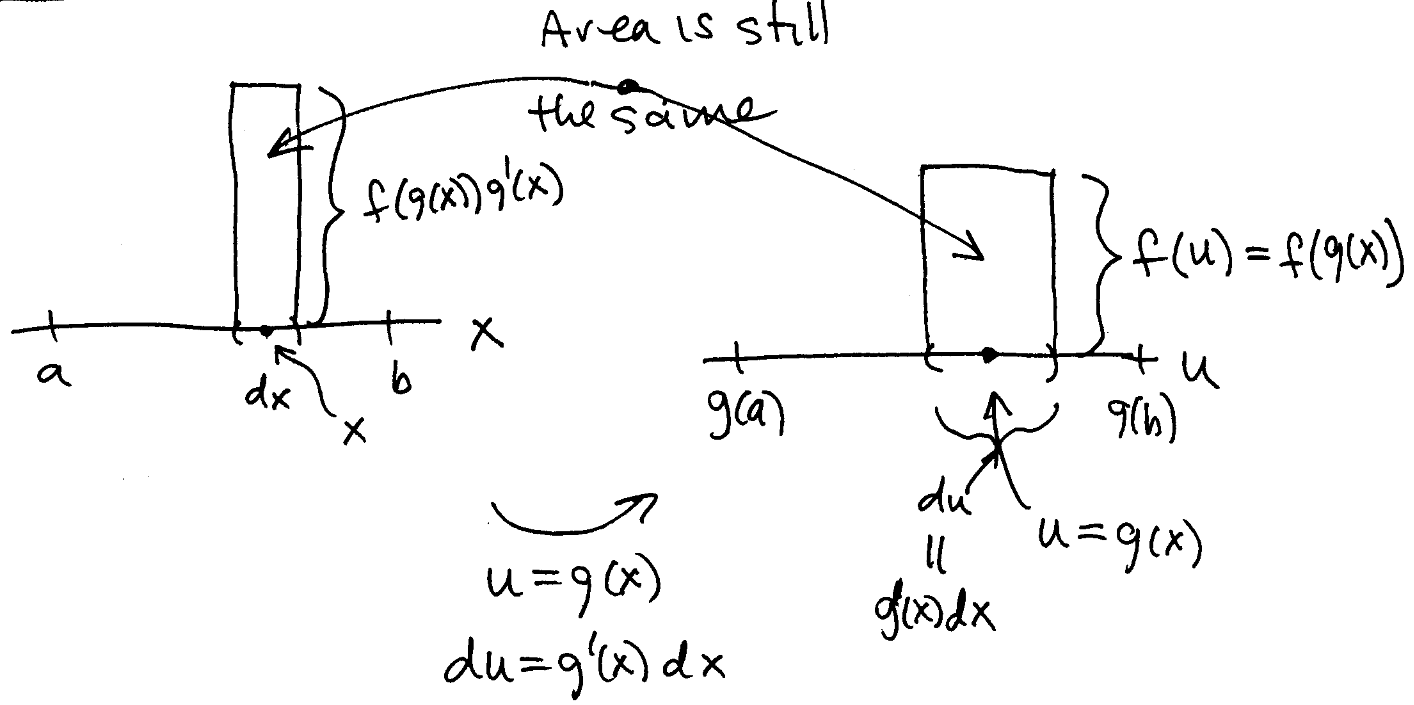
$$\int \underbrace{f(g(x))}_{f(u)}\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$

$u = g(x)$

$du = g'(x)\,dx$

What if I look at:

$$\boxed{\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$



e.g. $\displaystyle\int_0^1 x(1-x^2)^{1/2}\,dx$

$u = 1-x^2$

$du = -2x\,dx$

$x=0 \rightarrow u=1$

$x=1 \rightarrow u=0$

$$-\frac{1}{2}\int_0^1 \underbrace{-2x}\,(\overbrace{1-x^2}^{u})^{1/2}\,\underbrace{dx}$$

du

$$= -\frac{1}{2}\int_1^0 u^{1/2}\,du = \frac{1}{2}\int_0^1 u^{1/2}\,du$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_{u=0}^{u=1} = \frac{1}{3}u^{3/2}\Big|_{u=0}^{u=1}$$

$$= \frac{1}{3}\left(1^{3/2} - 0^{3/2}\right) = \frac{1}{3}$$

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e.g. $\displaystyle\int_2^4 \underset{\sim}{x}\,(\underbrace{x+1}_{u})^{1/2}\,\underbrace{dx}_{du}$

$u = x+1 \quad x = u-1$

$du = dx$

$x=2 \rightarrow u=3$

$x=4 \rightarrow u=5$

$$= \int_3^5 (u-1)\,u^{1/2}\,du$$

$$= \int_3^5 \left(u^{3/2} - u^{1/2}\right)du = \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_{u=3}^{u=5}$$

$$= \left(\frac{2}{5}(5)^{5/2} - \frac{2}{3}(5)^{3/2}\right) - \left(\frac{2}{5}(3)^{5/2} - \frac{2}{3}(3)^{3/2}\right) \quad \underline{\text{ok}}$$

e.g.

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3+2\cos(x)}\,dx$$

$\cancel{u=\sin(x)}$

$\cancel{du = \cos(x)\,dx}$

$u = 3+2\cos(x)$

$du = -2\sin(x)\,dx$

$$\boxed{\frac{du}{dx} = -2\sin(x)}$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$-\int \frac{\overbrace{-\sin(x)\,dx}^{du}}{\underbrace{3+2\cos(x)}_{3+2u}} = -\int \frac{du}{3+2u}$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{-2\sin(x)}{3+2\cos(x)}\,dx = -\frac{1}{2}\int_5^3 \frac{du}{u} = \frac{1}{2}\int_3^5 \frac{1}{u}\,du$$

$x=0 \rightarrow u = 3+2\cos(0) = 5$

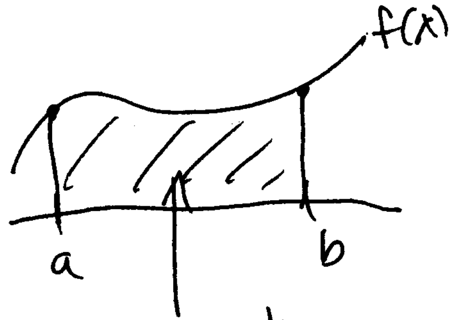
$x=\frac{\pi}{2} \rightarrow u = 3+2\cos\left(\frac{\pi}{2}\right) = 3$

$$= \frac{1}{2}\ln(u)\Big|_{u=3}^{5}$$

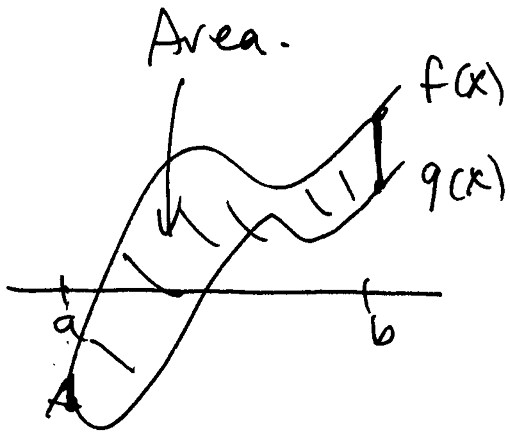
$$= \frac{1}{2}\ln(5) - \frac{1}{2}\ln(3) \,//$$

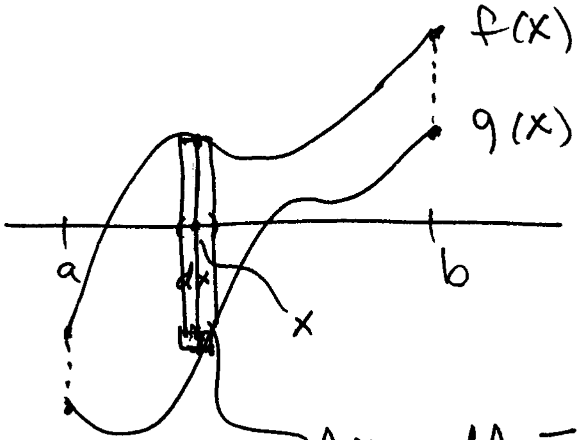
$$= \frac{1}{2}\ln\left(\frac{5}{3}\right) //$$

② Areas between curves.



Area $= \int_a^b f(x)\,dx$

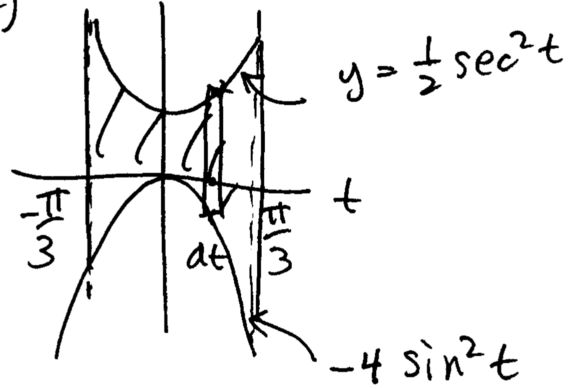




$$\text{Area} = dA = \left(f(x)-g(x)\right)\,dx$$

$$A = \int_a^b dA = \int_a^b \left(f(x)-g(x)\right)dx$$

eg #52)



$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2}\sec^2(t) - \left(-4\sin^2(t)\right)\right)dt$$

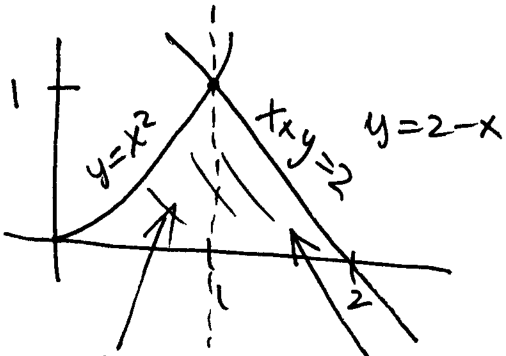
$$\sin^2(t) = \frac{1-\cos(2t)}{2}$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\sec^2(t) + 4\sin^2(t)\,dt \qquad 4\sin^2(t) = \cancel{4}^{2}\left(\frac{1-\cos(2t)}{\cancel{2}}\right)$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2}\sec^2(t) + 2 - 2\cos(2t)\right)dt$$

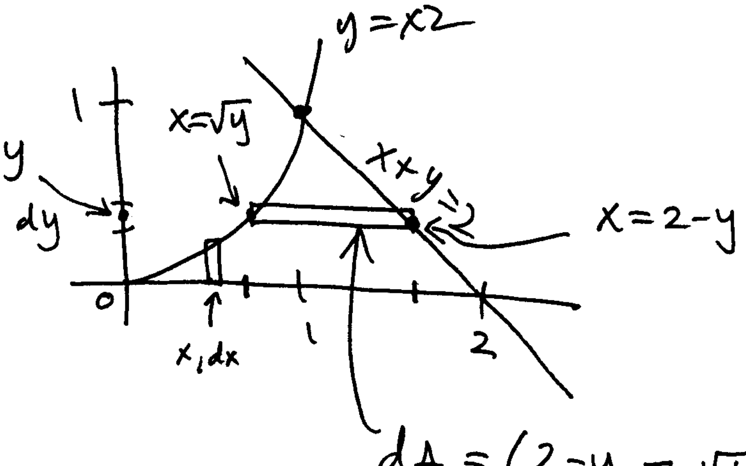
etc....

#58)



$$A = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx \quad \text{etc----}$$

Another way:



$$dA = (2-y-\sqrt{y}\,)dy$$

$$A = \int_0^1 (2-y-y^{1/2})\,dy$$