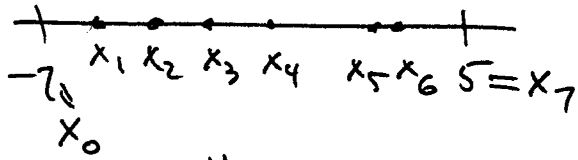
Example #3, 5.3

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k)\Delta x_k$$

suggests that the partition is getting finer and finer i.e. more and smaller subintervals.



Here $n = 7$

P is a partition of $[-7, 5]$

$P = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$

$\|P\|$ = size of partition $\stackrel{\text{DEF}}{=}$ length of longest subinterval in P

$$= \max_{1\le k\le n} \Delta x_k = \max_{1\le k\le 1} (x_k - x_{k-1})$$

Think of absolute value
$|x|$ = the size of x
or magnitude of x

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{(c_k^2 - 3c_k)}_{f(c_k)}\Delta x_k = \int_{-7}^{5} (x^2 - 3x)\,dx$$

MAPLE #3 due Tomorrow

Exam 4 Friday 4.5 – 5.5

Final Exam: TUESDAY JUNE 19

5.4 Fundamental Theorem of Calculus

Idea: How to calculate $\int_a^b f(x)\,dx$

We have seen the principle:

$$\left\{\begin{array}{l}\text{Area under the} \\ \text{curve } y=f(x) \\ \text{for } a \le x \le b\end{array}\right\} = \left\{\begin{array}{l}\text{Total displacement} \\ \text{of } \cancel{\text{the}}\text{ an antiderivative} \\ \text{of } f(x) \text{ on } [a,b]\end{array}\right\}$$

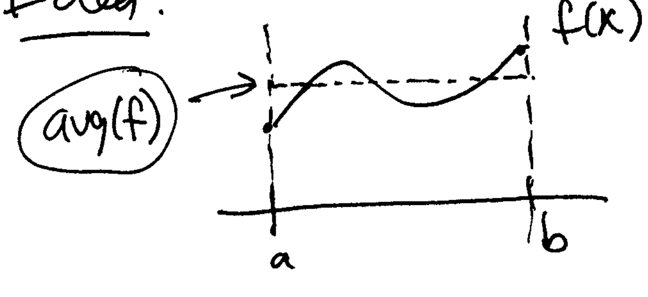
The Fund. Thm. is this principle in slightly different form.

Average value:

Def: The average value of $f(x)$ on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \text{avg}(f)$$

Idea:



Q: What constant can I replace $f(x)$ by to give same area under curve?

$$\text{Area under rectangle} = (b-a)\cdot \text{avg}(f)$$
$$\text{Area under curve} = \int_a^b f(x)\,dx$$

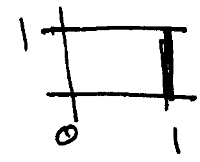
Then

$$\text{avg}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example Find average value of $3x^2+1$ on $[0,1]$

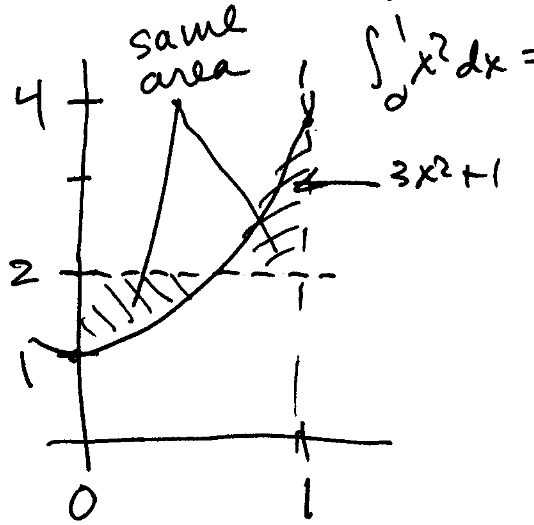
$$\text{avg} = \frac{1}{1-0}\int_0^1 (3x^2+1)\,dx = 3\cdot\frac{1}{3}+1 = 2$$

$$\int_0^1 3x^2+1\,dx = 3\underbrace{\int_0^1 x^2\,dx}_{\text{p. 378}} + \underbrace{\int_0^1 1\,dx}_{1}$$



$$\int_a^b x^2\,dx = \frac{b^3}{3}-\frac{a^3}{3}$$

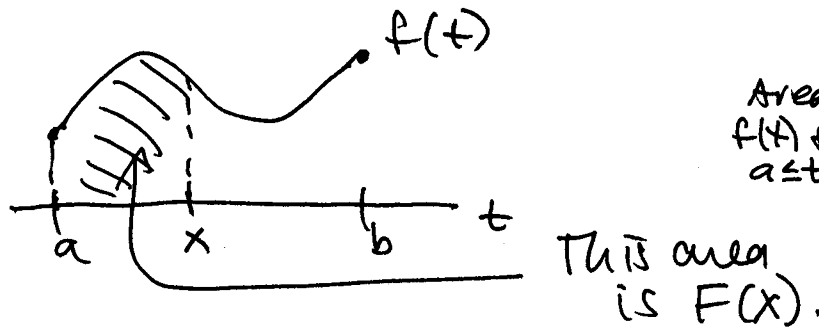
$$\int_0^1 x^2\,dx = \frac{1^3}{3}-\frac{0^3}{3} = \frac{1}{3}$$

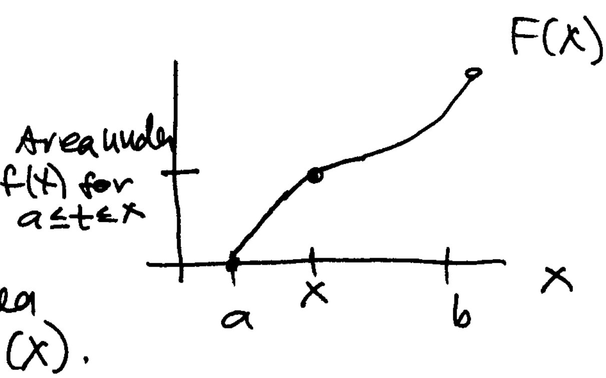


Fundamental Theorem Part I.

Given $f(x)$ on $[a,b]$ define

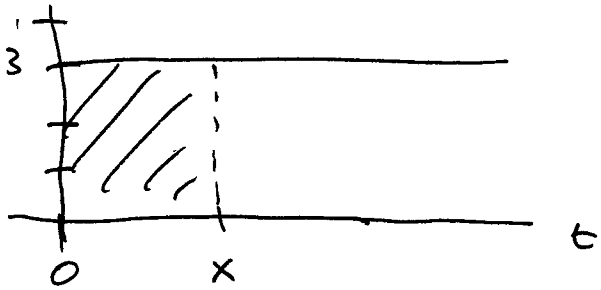
$F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$.



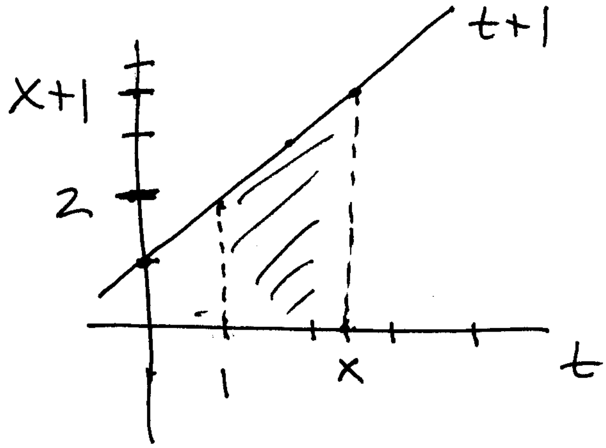


e.g. $F(a) = \int_a^a f(t)\,dt = 0$

e.g. $\int_0^x 3\,dt = 3x$



$\frac{d}{dx}(3x) = 3$



e.g. $\int_1^x (t+1)\,dt$

$= \begin{pmatrix}\text{avg}\\ \text{height}\end{pmatrix} \times (\text{width})$

$= \left(\frac{x+1+2}{2}\right)(x-1)$

$= \left(\frac{x+3}{2}\right)(x-1)$ = area under graph of $f(t) = t+1$ for $1 \le t \le x$.

$\frac{d}{dx}\left(\left(\frac{x+3}{2}\right)(x-1)\right) = \frac{1}{2}\frac{d}{dx}\left((x+3)(x-1)\right)$

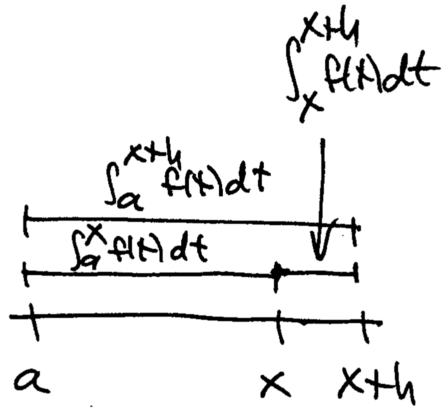
$= \frac{1}{2}\frac{d}{dx}\left(x^2+2x-3\right) = \frac{1}{2}(2x+2) = x+1$

Why is this true?

$$F(x)=\int_a^x f(t)\,dt. \quad \text{Verify } \underline{F'(x)=f(x)}.$$

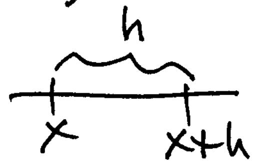
$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(F(x+h)-F(x)\right)$$

$$= \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right)$$



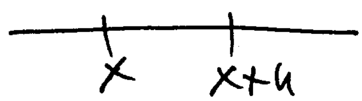
$$= \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_x^{a} f(t)\,dt\right)$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

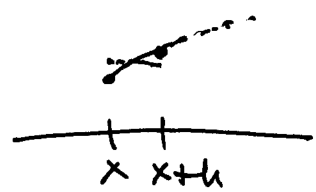


= avg value of f on $[x, x+h]$.

Idea: As $h \to 0$, avg value of f on $[x, x+h]$ converges to $f(x)$



$$\therefore \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$



$$\begin{array}{ccc} \| & & \| \\ F'(x) & = & f(x). \end{array}$$

F.T.C. Part II

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

Why? Let F be any antiderivative of f.

Since by FTC Part I,

$$\frac{d}{dx}\Bigg(\underbrace{\int_a^x f(t)\,dt}_{\text{an a.d. of } f(x)}\Bigg) = f(x)$$

I know that $\displaystyle\int_a^x f(t)\,dt = F(x) + C$

some const. C.

What is C?

$$0 = \int_a^a f(t)\,dt = F(a) + C, \text{ so } C = -F(a).$$

Finally $\displaystyle\int_a^x f(t)\,dt = F(x) - F(a)$.

So if $x = b$ $\boxed{\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)}$

e.g. $\displaystyle\int_0^1 (1-x^2)\,dx = x - \frac{1}{3}x^3 \Big|_{x=0}^{x=1}$

$= \left(1 - \frac{1}{3}(1)^3\right) - \left(0 - \frac{1}{3}(0)^3\right) = 1 - \frac{1}{3} - 0 = \frac{2}{3}.$

e.g. $\displaystyle\int_0^{2\pi} \cos(x)\,dx = \sin(x)\Big|_{x=0}^{x=2\pi} = \sin(2\pi) - \sin(0)$

$= 0 - 0 = 0.$

e.g. $\displaystyle\int_0^3 (-2x+4)\,dx = -\cancel{2}\cdot\frac{1}{\cancel{2}}x^2 + 4x \Big|_{x=0}^{x=3}$

$= -x^2 + 4x \Big|_{x=0}^{x=3} = \left(-(3)^2 + 4(3)\right) - \left(-0^2 + 4\cdot 0\right)$

$= -9 + 12 = 3.$

e.g. $\displaystyle\int_{-2}^2 (x^3 - 2x + 3)\,dx = \frac{1}{4}x^4 - x^2 + 3x \Big|_{x=-2}^{x=2}$

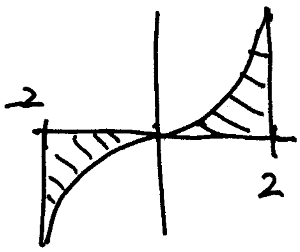
$= \left(\frac{1}{4}(2)^4 - (2)^2 + 3(2)\right) - \left(\frac{1}{4}(-2)^4 - (-2)^2 + 3(-2)\right)$

$= (\cancel{4} - \cancel{4} + 6) - (\cancel{4} - \cancel{4} - 6) = 6 - (-6) = 12$

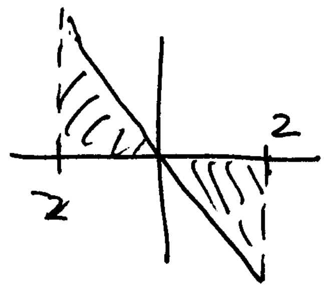
$$\int_{-2}^{2} x^3\,dx = \frac{1}{4}x^4 \Big|_{x=-2}^{x=2} = \frac{1}{4}(2)^4 - \frac{1}{4}(-2)^4 = 0$$

odd

$$\int_{-2}^{2} -2x\,dx = 0$$



x^3 is odd



$-2x$ is odd

<u>Fact</u>: Integral of odd function over a symmetric interval is zero.

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{if } f \text{ is odd.}$$

<u>eg</u>

$$\int_{1}^{2}\left(\frac{1}{x} + e^{-x}\right)dx = \ln(x) - e^{-x}\Big|_{x=1}^{x=2}$$

$$= \left(\ln(2) - e^{-2}\right) - \left(\cancelto{0}{\ln(1)} - e^{-1}\right)$$

$$= \ln(2) - e^{-2} + e^{-1} \;//$$

<u>e.g.</u>

$$\int_{9}^{4} \frac{1-\sqrt{u}}{\sqrt{u}}\,du = \int_{9}^{4} \frac{1-u^{1/2}}{u^{1/2}}\,du = \int_{9}^{4} (1-u^{1/2})\,u^{-1/2}\,du$$

$$\int_9^4 (u^{-1/2} - 1)\, du = \frac{1}{-\frac{1}{2}+1} u^{-\frac{1}{2}+1} - u \Big|_{u=9}^{u=4}$$

$$= 2u^{1/2} - u \Big|_{u=9}^{u=4} = \left(2(4)^{1/2} - (4)\right) - \left(2 \cdot 9^{1/2} - 9\right)$$

$$= (4-4) - (6-9) = 0 - (-3) = 3 \,//.$$

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eg #36 $\frac{d}{dx}\left(\int_1^{\sin(x)} 3t^2\, dt\right) = \frac{d}{dx}\left(t^3 \Big|_{t=1}^{t=\sin(x)} \right)$

$$= \frac{d}{dx}\left(\sin^3(x) - 1\right) = 3\sin^2(x) \cdot \cos(x)$$

Use FTC:

$$\frac{d}{dx}\left(\int_1^{\sin(x)} 3t^2\, dt\right) = \frac{d}{dx}\left(F(\sin(x)) - F(1)\right)$$

(where $F'(x) = 3x^2$)

$$= F'(\sin(x)) \cdot \cos(x) = 3\sin^2(x) \cdot \cos(x).$$

eg #44) $y = \int_0^{x^2} \cos(\sqrt{t})\, dt$

Find $\frac{dy}{dx}$. By FTC II,

$$y = \int_0^{x^2} \cos(\sqrt{t})\,dt = F(x^2) - F(0)$$

where $F'(x) = \cos(\sqrt{x})$

$$\frac{dy}{dx} = \frac{d}{dx}\Big(\underbrace{F(x^2)}_{\uparrow\ \text{chain rule}} - \underbrace{F(0)}_{\uparrow\ \text{const.}}\Big) = F'(x^2)(2x)$$

$$= \cos(\sqrt{x^2})(2x) = \cos(|x|) \cdot 2x.$$

5.5 Substitution rule.

So far we can find $\int f(x)dx$ only when $f(x)$ is a simple derivative.

But what about something like

$\int x(x^2+1)^{1/2}dx$? Cant do it.

How to approach this?

$\frac{1}{2}\int \underbrace{(x^2+1)}_{u}{}^{1/2}\underbrace{(2x)}_{du/dx}\,dx$ Looks like chain rule!

Substitution allows us to ~~recg~~ recognize when a function might be the result of chain rule.

If we write $u = x^2+1$

Then $\frac{du}{dx} = 2x$ so integral becomes

$$\frac{1}{2}\int u^{1/2}\cdot\frac{du}{\cancel{dx}}\cdot\cancel{dx} = \frac{1}{2}\int u^{1/2}du$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2}+c = \frac{1}{3}u^{3/2}+c = \frac{1}{3}(x^2+1)^{3/2}+c$$

Check: $\frac{d}{dx}\left(\frac{1}{3}(x^2+1)^{3/2}+c\right) = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(x^2+1)^{1/2}(\cancel{2}x)$

$= x(x^2+1)^{1/2}$ ✓

"substitution" also called "change of variables".

eg

$$\int (4t-1)^{1/2}\,dt \qquad u = 4t-1$$

$$\frac{du}{dt} = 4$$

$$= \frac{1}{4}\int \underbrace{(4t-1)}_{u}^{1/2}\underbrace{(4)}_{\frac{du}{dt}}\,dt$$

$$= \frac{1}{4}\int u^{1/2}\,\frac{du}{dt}\cdot dt = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{\cancel{4}_{2}}\cdot\frac{\cancel{2}}{3}u^{3/2} + C$$

$$= \frac{1}{6}u^{3/2} + C = \frac{1}{6}(4t-1)^{3/2} + C$$

eg

$$\int x\,\sin(2x^2)\,dx \qquad u = 2x^2$$

$$\frac{du}{dx} = 4x$$

$$\frac{1}{4}\int \sin(2x^2)\,(4x)\,dx = \frac{1}{4}\int \sin(u)\,\frac{du}{dx}\,dx$$

$$= \frac{1}{4}\int \sin(u)\,du = \frac{1}{4}(-\cos(u)) + C$$

$$= -\frac{1}{4}\cos(2x^2) + C \quad //$$

e.g.

$$\int \frac{4y}{\sqrt{2y^2+1}}\,dy$$

$u = 2y^2+1$

$\frac{du}{dy} = 4y$

small shortcut:

$u = 2y^2+1$

$du = 4y\,dy$

$$\int \frac{\overbrace{4y\,dy}^{du}}{\sqrt{\underbrace{2y^2+1}_{u}}}$$

$$= \int \frac{1}{\sqrt{u}}\,du$$

$$= \int \frac{1}{\sqrt{2y^2+1}}(4y)\,dy$$

$$= \int \frac{1}{\sqrt{u}}\frac{du}{dy}\cdot dy = \int \frac{1}{\sqrt{u}}\,du$$

$$= \int u^{-1/2}\,du = 2u^{1/2} + C$$

$$= 2(2y^2+1)^{1/2} + C$$

e.g.

$$\int \frac{\overbrace{\ln(t)}^{u}}{t}\,dt\;\; du$$

$u = \ln(t)$

$du = \frac{1}{t}\,dt \quad \left(\frac{du}{dt} = \frac{1}{t}\right)$

$$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(t))^2 + C$$

e.g.

$$\int \frac{d\theta}{\sqrt{e^{2\theta}-1}}$$

$u = e^{2\theta}-1 \longrightarrow e^{2\theta} = u+1$

$du = 2\boxed{e^{2\theta}}\,d\theta$

$du = 2(u+1)\,d\theta$

$d\theta = \frac{du}{2(u+1)}$

$$= \int \frac{du}{2(u+1)}\cdot\frac{1}{\sqrt{u}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}(u+1)}$$ NO HELP

Try something else. $u = e^{\theta}$

$$du = e^{\theta}\, d\theta = u\, d\theta$$

$$d\theta = \frac{du}{u}$$

$$\int \frac{d\theta}{\sqrt{e^{2\theta}-1}} = \int \frac{du}{u} \cdot \frac{1}{\sqrt{u^2-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}(u) + c$$

$$= \sec^{-1}(e^{\theta}) + c \quad //$$

Calculator says: $\int \frac{d\theta}{\sqrt{e^{2\theta}-1}} = \tan^{-1}((e^{2\theta}-1)^{1/2}) + c$

$$\frac{d}{dx} \tan^{-1}((e^{2\theta}-1)^{1/2}) = \frac{1}{(\cancel{e^{2\theta}} \cancel{-1}) \cancel{+1}} \cdot \cancel{\tfrac{1}{2}} (e^{2\theta}-1)^{-1/2} \cdot \cancel{2} \cancel{e^{2\theta}}$$

$$= (e^{2\theta}-1)^{-1/2} = \frac{1}{\sqrt{e^{2\theta}-1}}$$

One more time!

$$\int \frac{d\theta}{\sqrt{e^{2\theta}-1}}$$

$u = (e^{2\theta}-1)^{1/2}$ $\quad u^2 = e^{2\theta}-1$ $\quad \boxed{e^{2\theta} = u^2+1}$

$$du = \cancel{\tfrac{1}{2}}(e^{2\theta}-1)^{-1/2}(\cancel{2}e^{2\theta})\, d\theta$$

$$= \frac{e^{2\theta}}{\sqrt{e^{2\theta}-1}}\, d\theta$$

$$= \frac{u^2+1}{u}\, d\theta$$

$$d\theta = \frac{u}{u^2+1}\, du$$

$$= \int \frac{\cancel{u}}{u^2+1} \frac{1}{\cancel{u}}\, du$$

$$= \int \frac{du}{u^2+1} = \tan^{-1}(u) + c$$

$$= \tan^{-1}((e^{2\theta}-1)^{1/2}) + c.$$