Exam 4 Friday 4.5 – 5.5

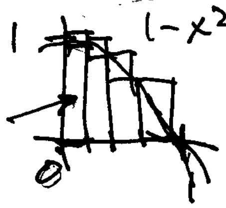
MAPLE #3 due Thursday

Final Exam $\boxed{\text{Tuesday } 6-19}$ NEW DATE!

Antiderivative: Given $f(x)$ find $F(x)$ so that $F'(x) = f(x)$.

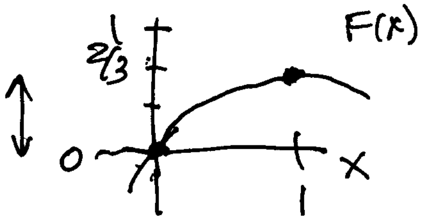
We write: $\boxed{\int f(x)\,dx = F(x) + C}$

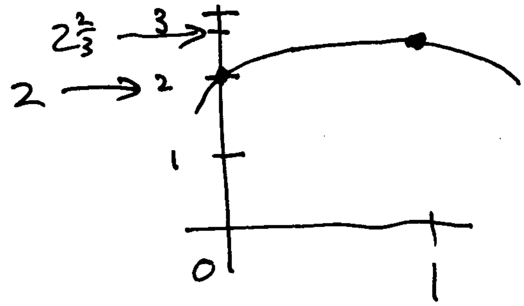
Area under a curve:



Principle: Area under the graph of $f(x)$ = Total change in antiderivative of $f(x)$.

$\underline{f(x) = 1 - x^2}$ $\underline{F(x) = x - \frac{1}{3}x^3}$ $\underline{\text{or}}$ $G(x) = x - \frac{1}{3}x^3 + 2$

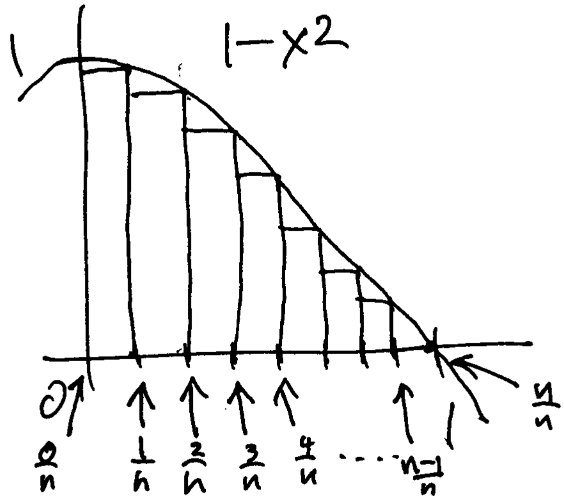




5.2 Sigma Notation

Recall our area problem:

Idea: Fill with rectangles in a particular way



① Divide $[0,1]$ into n equal subintervals

What are these subintervals?

$[0,\frac{1}{n}]\ [\frac{1}{n},\frac{2}{n}]\ [\frac{2}{n},\frac{3}{n}]\ [\frac{3}{n},\frac{4}{n}], \dots, [\frac{n-1}{n},1]$ n alltogether

Length of each is $1/n$.

② Form the <u>lower sum</u>

Use right endpoint ~~as~~ to give height of rectangle.

$$A \approx L_n = f\left(\tfrac{1}{n}\right)\tfrac{1}{n} + f\left(\tfrac{2}{n}\right)\cdot\tfrac{1}{n} + f\left(\tfrac{3}{n}\right)\cdot\tfrac{1}{n} + \cdots + f\left(\tfrac{n-1}{n}\right)\cdot\tfrac{1}{n} + f(1)\cdot\tfrac{1}{n}$$

$$= \frac{1}{n}\left[\left(1-\frac{1^2}{n^2}\right)+\left(1-\frac{2^2}{n^2}\right)+\left(1-\frac{3^2}{n^2}\right)+\cdots+\left(1-\frac{(n-1)^2}{n^2}\right)+\left(1-\frac{n^2}{n^2}\right)\right]$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k^2}{n^2}\right)$$

e.g. $$\sum_{k=1}^{5}\left(3k+\frac{1}{k}\right) = \underbrace{(3+1)}_{k=1} + \underbrace{\left(6+\tfrac{1}{2}\right)}_{k=2} + \underbrace{\left(9+\tfrac{1}{3}\right)}_{k=3} + \underbrace{\left(12+\tfrac{1}{4}\right)}_{k=4}$$

index

$$+ \underbrace{\left(15+\tfrac{1}{5}\right)}_{k=5} = 46 + \tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}$$

$$= 46 + \frac{30+20+15+12}{60}$$

$$= 46 + \frac{77}{60} = 47\tfrac{17}{60}$$

$$= \frac{1}{n}\left(\sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \frac{k^2}{n^2}\right)$$

$\uparrow$ common factor of $\frac{1}{n^2}$

$$= \frac{1}{n}\left(\sum_{k=1}^{n} 1 - \frac{1}{n^2}\sum_{k=1}^{n} k^2\right)$$

$$= \frac{1}{n}\left(n - \frac{1}{n^2}\sum_{k=1}^{n} k^2\right)$$

$$\sum_{k=1}^{5} 1 = 1+1+1+1+1 = 5$$

$$= 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{3} k^2 = 1^2+2^2+3^2 = 1+4+9 = 14$$

$$\sum_{k=1}^{5} k^2 = 1^2+2^2+3^2+4^2+5^2 = 1+4+9+16+25 = 55$$

Is there a formula for $\sum_{k=1}^{n} k^2$? Yes.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$n=3$: $\frac{\cancel{3}(\cancel{4}^{2})(7)}{\cancel{6}\,\cancel{2}} = 14$

$n=5$: $\frac{5(\cancel{6})(11)}{\cancel{6}} = 55$

$$= 1 - \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

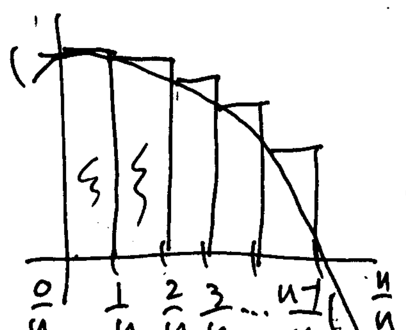
$$= 1 - \frac{n(n+1)(2n+1)}{6n^3} = L_n$$

Actual area should be

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} 1 - \frac{\overset{2n^3 + \square n^2 + \square n + \square}{n(n+1)(2n+1)}}{6n^3}$$

$$= \lim_{n\to\infty} 1 - \frac{2\cancel{n^3}}{6\cancel{n^3}} = 1 - \frac{1}{3} = \frac{2}{3}$$

If we do this for U_n = upper sum with n subintervals, we get same thing.



For U_n, use left endpoint

$$A \approx U_n = f\left(\tfrac{0}{n}\right)\cdot\tfrac{1}{n} + f\left(\tfrac{1}{n}\right)\cdot\tfrac{1}{n} + f\left(\tfrac{2}{n}\right)\cdot\tfrac{1}{n} + \cdots + f\left(\tfrac{n-1}{n}\right)\cdot\tfrac{1}{n}$$

$$= \frac{1}{n}\left[\left(1-\frac{0^2}{n^2}\right) + \left(1-\frac{1^2}{n^2}\right) + \cdots + \left(1-\frac{(n-1)^2}{n^2}\right)\right]$$

$$\sum_{k=0}^{5} 1 = 1+1+1+1+1+1 = 6$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\left(1-\frac{k^2}{n^2}\right)$$

$$= 1 - \frac{1}{n^3}\sum_{k=0}^{n-1} k^2$$

$$\sum_{k=0}^{n-1} k^2 = \underbrace{0}_{k=0} + \sum_{k=1}^{n-1} k^2 = \frac{(n-1)(n-1+1)(\overset{2n-2+1}{2(n-1)+1})}{6}$$

$$= \frac{(n-1)(n)(2n-1)}{6}$$

$$= 1 - \frac{1}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6} = U_n$$

$$A = \lim_{n\to\infty} U_n = \lim_{n\to\infty}\left(1 - \frac{\overset{2n^3+(\cdots)}{(n-1)(n)(2n-1)}}{6n^3}\right)$$

$$= 1 - \lim_{n\to\infty} \frac{2n^3}{6n^3} = 1 - \frac{1}{3} = \frac{2}{3} \ //$$

$$\therefore \quad L_n \le A \le U_n$$

$$\begin{array}{ccc} \downarrow & \| & \downarrow \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{array}$$

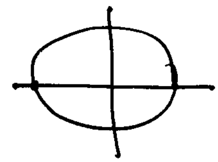
eg #2 p369

$$\sum_{k=1}^{3} \frac{k-1}{k} = \underbrace{\frac{1-1}{1}}_{k=1} + \underbrace{\frac{2-1}{2}}_{k=2} + \underbrace{\frac{3-1}{3}}_{k=3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

Also do: $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k} = \sum_{k=1}^{3} 1 - \frac{1}{k} = \sum_{k=1}^{3} 1 - \sum_{k=1}^{3} \frac{1}{k}$

$$= 3 - \left(1 + \frac{1}{2} + \frac{1}{3}\right) = 3 - 1 - \frac{1}{2} - \frac{1}{3} = 2 - \frac{5}{6} = \frac{7}{6}$$

#6 $\displaystyle\sum_{k=1}^{4} (-1)^k \cos(k\pi)$



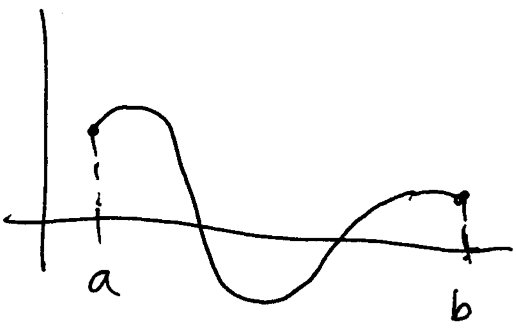
$$= \underbrace{(-1)\cos(\pi)}_{k=1} + \underbrace{(-1)^2\cos(2\pi)}_{k=2} + \underbrace{(-1)^3\cos(3\pi)}_{k=3} + \underbrace{(-1)^4\cos(4\pi)}_{k=4}$$

$$= (-1)(-1) + (1)(1) + (-1)(-1) + (1)(1)$$

$$= 1 + 1 + 1 + 1 = 4.$$

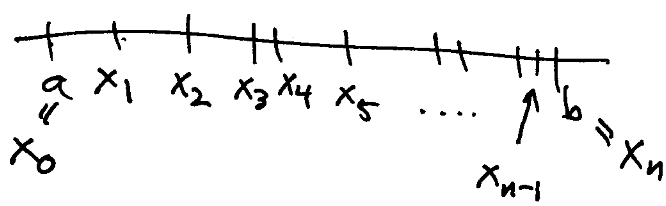
5.3 Definite Integral.

Addresses the general problem of finding "areas" under graphs of functions.



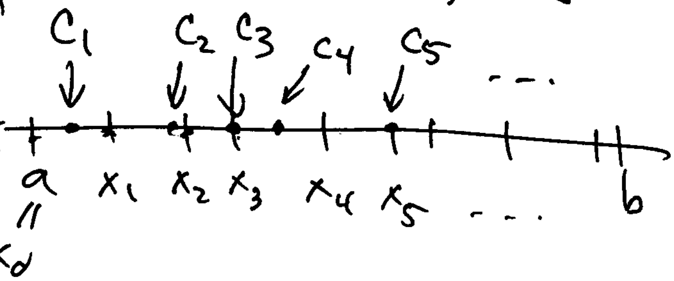
graph of $f(x)$.

① Divide $[a,b]$ into subintervals. This is called a partition of $[a,b]$ and is given by



$$P = \{x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b\}$$

② Choose a number c_k in $[x_{k-1}, x_k]$



③ Form the sum:

$$\underbrace{\sum_{k=1}^{n} \underbrace{f(c_k)}_{\text{"height" of rectangle}} \cdot \underbrace{\Delta x_k}_{\text{width of rectangle}}}_{\text{"AREA" of kth RECTANGLE}}$$

$$\Delta x_k = x_k - x_{k-1}$$

$$\Delta x_1 = x_1 - x_0$$

$$\Delta x_2 = x_2 - x_1$$

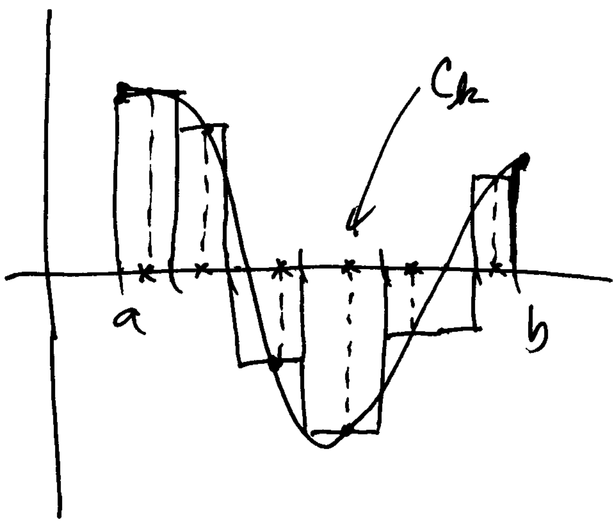
$$\Delta x_3 = x_3 - x_2$$

etc.

Note that if $f(c_k) < 0$ then the "area" of corresponding rectangle is negative.

The sum $\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$ is called a

RIEMANN SUM.



Note that there are negative and positive contributions to this Riemann sum.

(4) We expect that as $\Delta x_k \to 0$ that the Riemann sums will get closer to some fixed number, call it I.

We say

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

where $\|P\| = \max_{k=1,\dots,n} \Delta x_k$ = length of largest subinterval in P.

We write:

$$I = \int_a^b f(x)\,dx$$

(a) Intuitively:

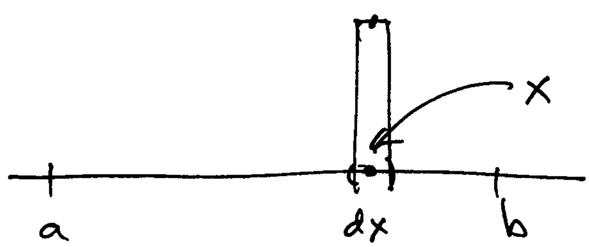
$$\sum_{k=1}^{n} f(c_k)\Delta x_k \longrightarrow \int_a^b f(x)\,dx$$

$$\sum_{k=1}^{n} \longrightarrow \int_a^b$$

$$f(c_k) \longrightarrow f(x)$$

$$\Delta x_k \longrightarrow dx$$

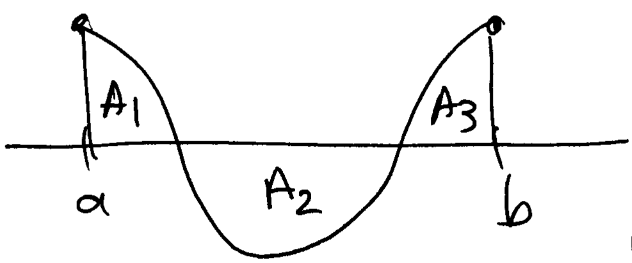
Think of this as width of an infinitesimal interval



$f(x)\cdot dx$ = area of an infinitesimal rectangle

$\int_a^b f(x)\,dx$ = sum of infinitesimal areas over all x in $[a,b]$.

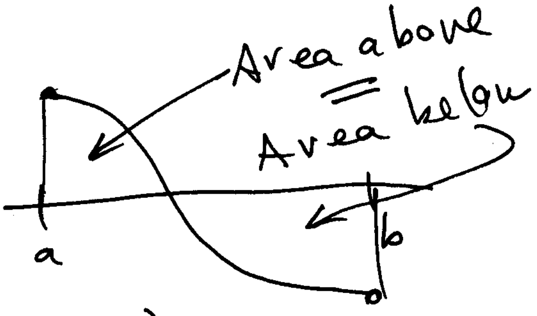
(b) In fact $\int_a^b f(x)\,dx$ is the "area" under the graph of f(x) understood to mean the (actual area above the x axis) — (actual area below the x axis)



A_1, A_2, A_3 are actual areas of regions. Then

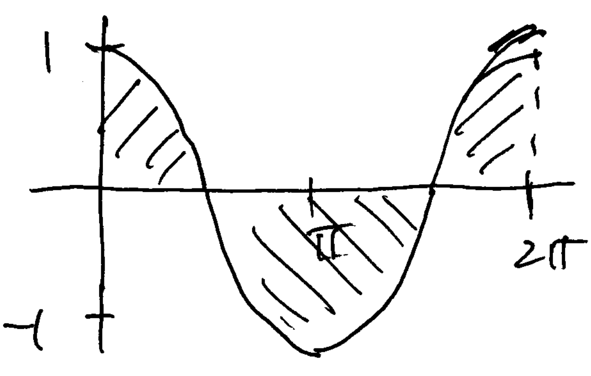
$$\int_a^b f(x)\,dx = A_1 + A_3 - A_2$$

This means we <u>can</u> have $\int_a^b f(x)\,dx = 0$ even if $f(x) \neq 0$.



then $\int_a^b f(x)\,dx = 0$.

e.g. cos(x)



Area (▨) = Area (▧)

so

$$\int_0^{2\pi} \cos(x)\,dx = 0.$$

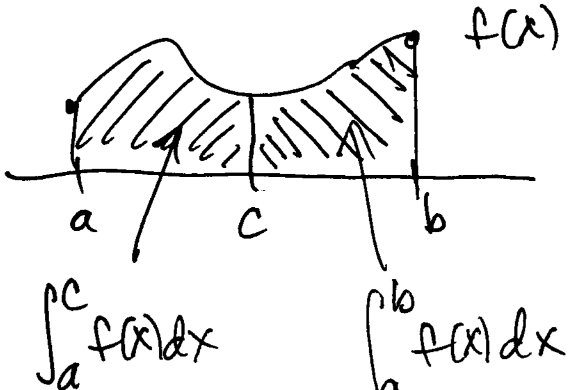
e.g. $\displaystyle\int_0^1 (1-x^2)\,dx = \frac{2}{3}$ (from before)

eg #2 $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 \,\Delta x_k = \int_{-1}^{0} 2x^3\,dx$

P partition of $[-1,0]$

Rules for definite integrals.

e.g. Rules #1, #5 on p 374



$\displaystyle\int_a^b f(x)\,dx$ = area under curve.

Then

$$\int_a^{c} f(x)\,dx + \int_{c}^b f(x)\,dx = \int_a^b f(x)\,dx$$

can be dropped.

Manipulate above equation:

$$\int_a^b f(x)\,dx - \int_c^b f(x)\,dx = \int_a^c f(x)\,dx$$

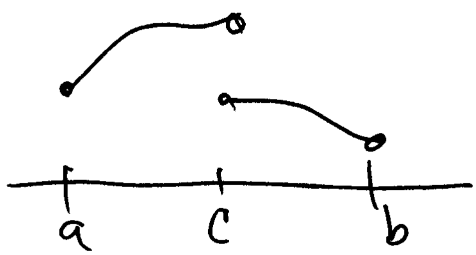
same as:

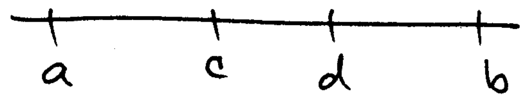
$$\int_a^{b} f(x)\,dx + \int_{b}^c f(x)\,dx = \int_a^c f(x)\,dx$$

So we need to say:

$$\int_b^c f(x)\,dx = -\int_c^b f(x)\,dx \qquad \text{Rule \#1}$$

$$\boxed{\int_b^c f(x)\,dx + \int_c^b f(x)\,dx = \int_b^b f(x)\,dx = 0}$$





$$\int_a^c f(x)\,dx + \int_d^b f(x)\,dx \neq \int_a^b f(x)\,dx$$

must be same

<u>e.g. #10</u> Spse

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4$$

(a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = (-2)(-1) = 2$

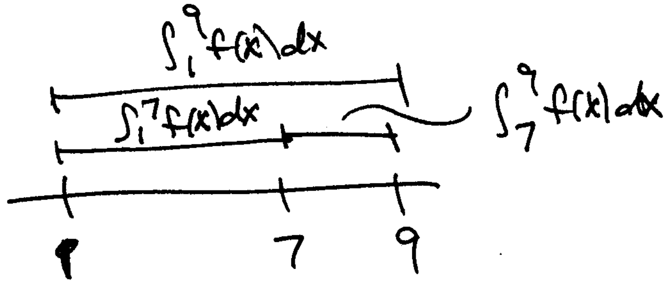
(b) $\int_7^9 (f(x)+h(x))\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5+4 = 9$

(c) $\int_7^9 (2f(x) - 3h(x))\,dx = \int_7^9 2f(x)\,dx - \int_7^9 3h(x)\,dx$

$= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2(5) - 3(4) = -2$ //

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

(e) $\int_1^7 f(x)\,dx$



$\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1 - 5 = -6$

(f) $\int_9^7 (h(x) - f(x))\,dx = -\int_7^9 (h(x) - f(x))\,dx$

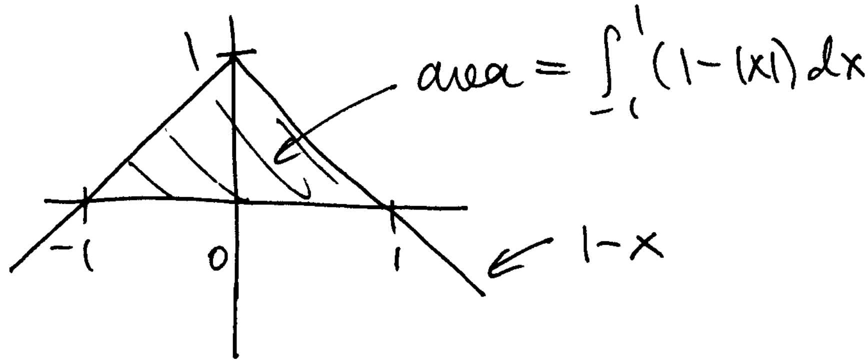
$= \int_7^9 (-h(x) + f(x))\,dx$

$= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx$

$= -4 + 5 = 1$ //

e.g. #20 $\int_{-1}^1 (1 - |x|)\,dx$

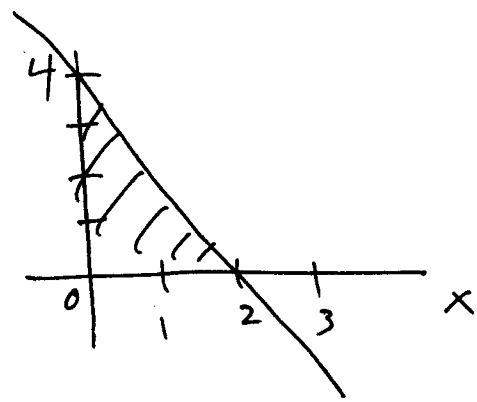
$$1-|x| = \begin{cases} 1-x & \text{if } x \geq 0 \\ 1+x & \text{if } x < 0 \end{cases}$$



$$\int_{-1}^{1} (1-|x|)\,dx = \frac{1}{2}(2)(1) = 1 \quad \text{(area of triangle)}$$

e.g. $\int_0^2 (-2x+4)\,dx$

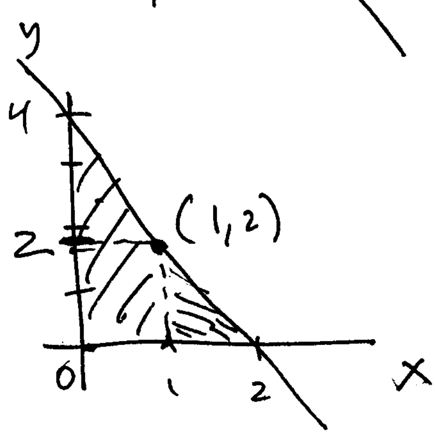
$= \frac{1}{2}(2)(4) = 4$



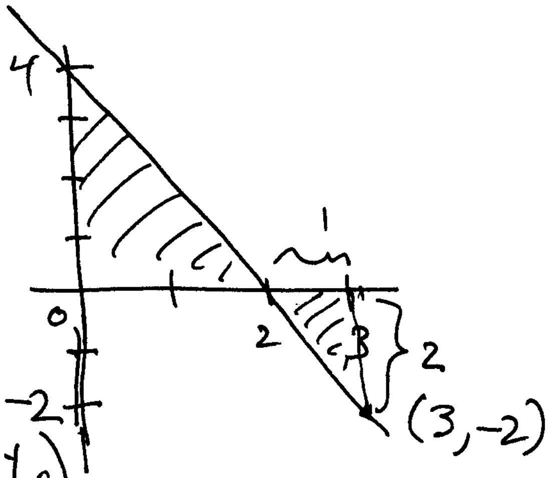
$\int_0^1 (-2x+4)\,dx$

$=$ (base)(average height)

$= (1)\left(\frac{2+4}{2}\right) = 3$



$$\int_0^3 (-2x+4)\,dx$$



= (area of large triangle)

− (area of small triangle)

$$= \frac{1}{2}(2)(4) - \frac{1}{2}(1)(2) = 4 - 1 = 3$$ //