Exam 4 Friday
MAPLE 3 due Thursday
Final Exam – June 19 $9^{\underline{30}} - 12^{\underline{0}}$

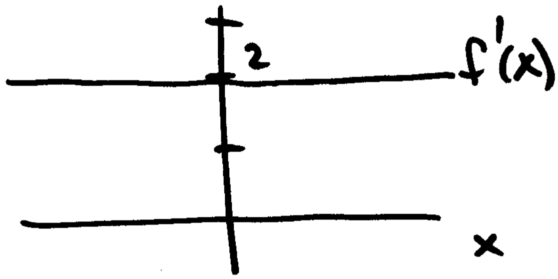
4.8 Antiderivative

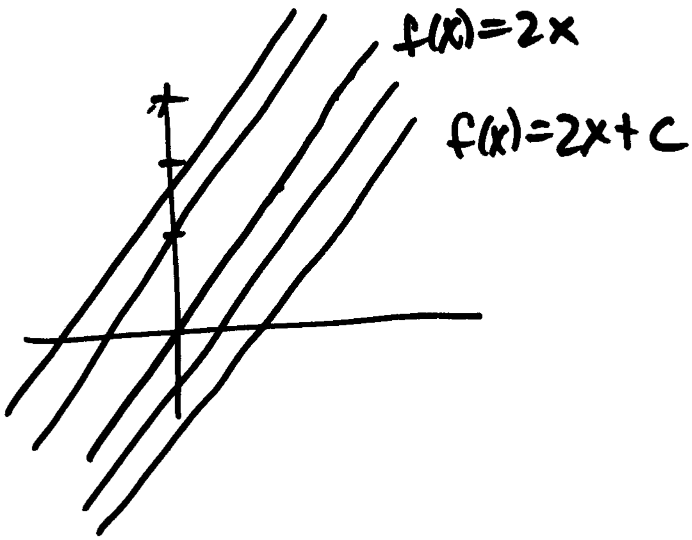
Idea: Given $f'(x)$ can we find $f(x)$?

e.g. $f'(x) = 2$

$f(x) = 2x$ a solution
$f(x) = 2x+1$ "
$f(x) = 2x-5$ "

In general the solution is $f(x) = 2x + C$ where C is constant.



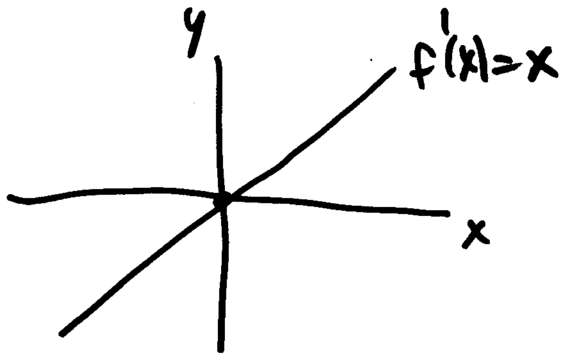


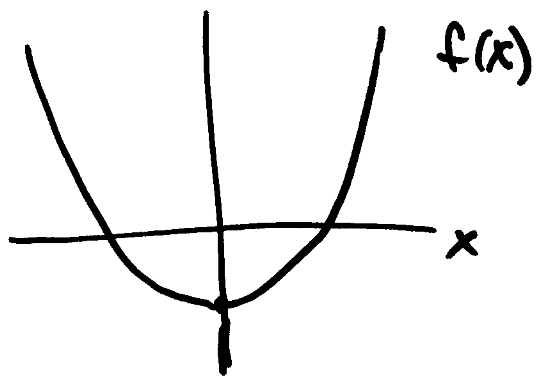
e.g. $f'(x) = x$ $\quad$ $f(x) = \frac{1}{2}x^2$ a solution

$f(x) = \frac{1}{2}x^2 + \frac{5}{3}$ "

$f(x) = \frac{1}{2}x^2 + \ln(2)$ "

In general $f(x) = \frac{1}{2}x^2 + C$, C const.





e.g. $f'(x) = 3^x$ $\quad$ $f(x) = \frac{1}{\ln(3)}3^x$ ✓

$\cancel{\frac{d}{dx}3^x = x3^{x-1}}$ $\quad$ $\cancel{\frac{d}{dx}e^x = xe^{x-1}}$

$\frac{d}{dx}e^x = e^x$ $\quad$ $\frac{d}{dx}3^x = \ln(3)\cdot 3^x$

$f'(x) = \cancel{\ln(3)}\cdot 3^x\cdot\frac{1}{\cancel{\ln(3)}}$
$= 3^x$

In general $f(x) = \frac{1}{\ln(3)}\cdot 3^x + C$, C const.

<u>Def</u>: A function $F(x)$ is <u>an</u> <u>antiderivative</u> of $f(x)$ if $F'(x) = f(x)$.

<u>Thm</u>: If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ then $F(x) = G(x) + C$.

Antiderivative rules (p.333)

<u>e.g.</u>

Function	Gen. antiderivative
$\sin(kx)$	$-\frac{1}{k}\cos(kx) + C$

$f(x) = \sin(kx)$

1st guess:

$$F(x) = \cos(kx)$$

$$\xrightarrow{\frac{d}{dx}} -\sin(kx) \cdot k$$

2nd guess:

$$F(x) = \frac{-1}{k}\cos(kx)$$

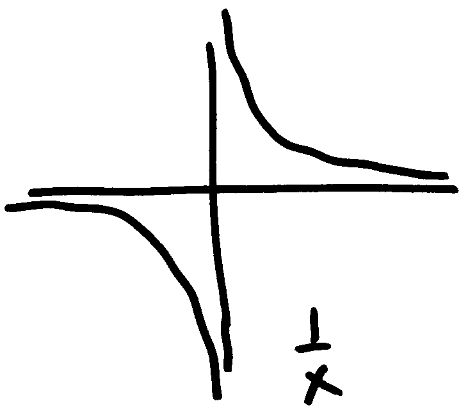
$$\xrightarrow{\frac{d}{dx}} +\frac{1}{\cancel{k}}(+\sin(kx) \cdot \cancel{k}) = \sin(kx) \checkmark$$

$f(x) = x^r \qquad F(x) = \frac{1}{r+1}x^{r+1} + C$

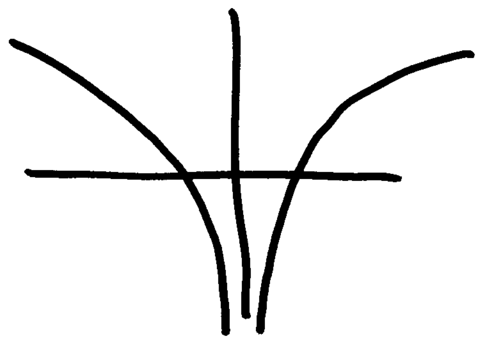
$r \neq -1$

$f(x) = x^{-1} = \frac{1}{x} \qquad F(x) = \ln(x) + c$

(In fact, $F(x) = \ln|x| + c$)



$\frac{1}{x}$



$f(x) = \sin(x) \qquad F(x) = -\cos(x) + C$

$f(x) = \cos(x) \qquad F(x) = \sin(x) + C$

<u>e.g.</u> $f(x) = 2x^{-3} + 2 = \frac{2}{x^3} + 2$

$F(x) = 2\left(\frac{1}{-2}x^{-2}\right) + 2x + C$

$= -x^{-2} + 2x + C.$

Can think:
$2 = 2x^0 \xrightarrow{A.D.} 2 \cdot \frac{1}{1}x^1 = 2x$

e.g. $f(x)=\frac{1}{3x}+\cos(5x)=\frac{1}{3}\cdot\frac{1}{x}+\cos(5x)$

$$F(x)=\frac{1}{3}(\ln(x))+\frac{1}{5}\sin(5x)+C$$

$$f(x)=\frac{1}{3x}\,,\quad \cancel{F(x)=\ln(3x)} \xrightarrow{\frac{d}{dx}} \frac{1}{3x}\cdot 3=\frac{1}{x}$$

$$f(x)=\frac{1}{x^3}\qquad \cancel{F(x)=\ln(x^3)}=3\ln(x)$$

$$\downarrow \tfrac{d}{dx}\qquad\qquad \downarrow \tfrac{d}{dx}$$

$$\frac{1}{x^{\cancel{3}}}\cdot 3\cancel{x^2}=\frac{3}{x}\qquad 3\cdot\frac{1}{x}=\frac{3}{x}$$

e.g. $f(x)=\frac{1}{x^3}=x^{-3}$ $\quad F(x)=\frac{-1}{2}x^{-2}+C$

$$F'(x)=\frac{+1}{\cancel{2}}(+\cancel{2}x^{-3})=x^{-3}\ \checkmark$$

e.g.

$$f(x) = \sec^2(3x) + e^{x/2}$$

$$F(x) = \frac{1}{3}\tan(3x) + 2e^{x/2} + C$$

$$\Big\downarrow \frac{d}{dx}$$

$$2e^{x/2}\cdot\frac{1}{2} = e^{x/2} \quad \underline{\text{ok}}$$

Indefinite integral

To denote a general antiderivative we use "integral notation":

$$\int f(x)\,dx = F(x) + c \quad \text{where } F'(x) = f(x)$$

e.g. $\int (2x^{-3} + 2)\,dx = -x^{-2} + 2x + c$

e.g. $\int \left(\frac{5}{2x} + \sin(\pi x)\right)dx$

$$= \frac{5}{2}\ln(x) - \frac{1}{\pi}\cos(\pi x) + C$$

Initial value problem.

$$\frac{dy}{dx} = f(x) \quad y(x_0) = y_0. \quad \text{Find } y.$$

given

$(y = y(x))$

e.g. $\frac{dy}{dx} = 9x^2 - 4x + 5 \quad \underline{y(-1) = 0}$

$$y = y(x) = \int (9x^2 - 4x + 5)\,dx$$

$$= 9 \cdot \frac{1}{3}x^3 - 4 \cdot \frac{1}{2}x^2 + 5x + c$$

$$= 3x^3 - 2x^2 + 5x + c$$

$y(-1) = 0$: initial value

use it to find c.

$$0 = y(-1) = 3(-1)^3 - 2(-1)^2 + 5(-1) + c$$

$$= -3 - 2 - 5 + c$$

$$= -10 + c$$

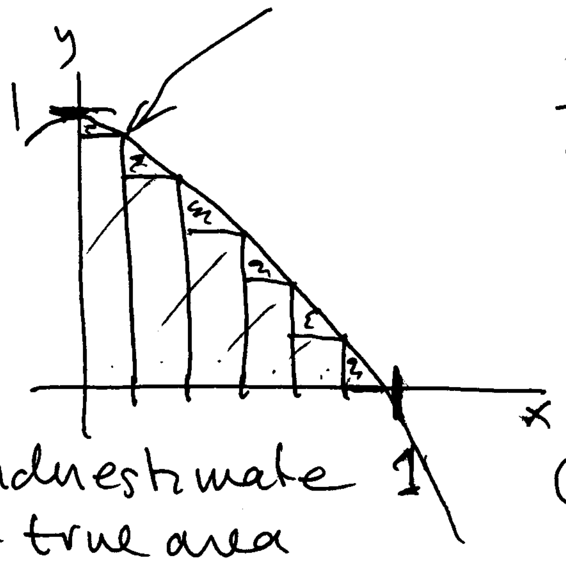
$\therefore c = 10. \qquad y = 3x^3 - 2x^2 + 5x + 10$ //

5.1 Estimating Areas with Finite Sums.

Problem:

Find the area under graph of $f(x)$ between $x=a$ and $x=b$.

e.g. $f(x) = 1 - x^2 \qquad 0 \le x \le 1$



Underestimate of true area

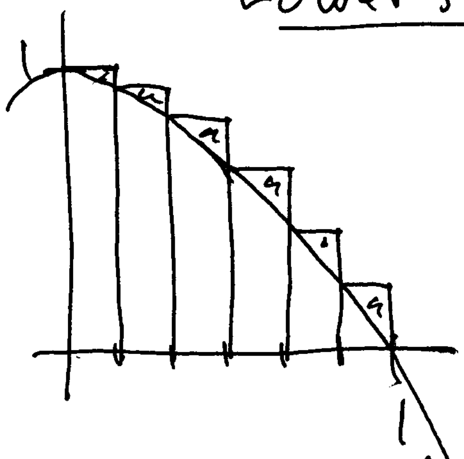
Lower sum

Idea: Approximate the region by a collection of simple regions whose area is known, eg. by rectangles, i.e.

① Divide $[0,1]$ into equal subintervals (6 in picture)

② Draw a rectangle over each subinterval approximating area under curve.

③ Add up areas of rectangles.

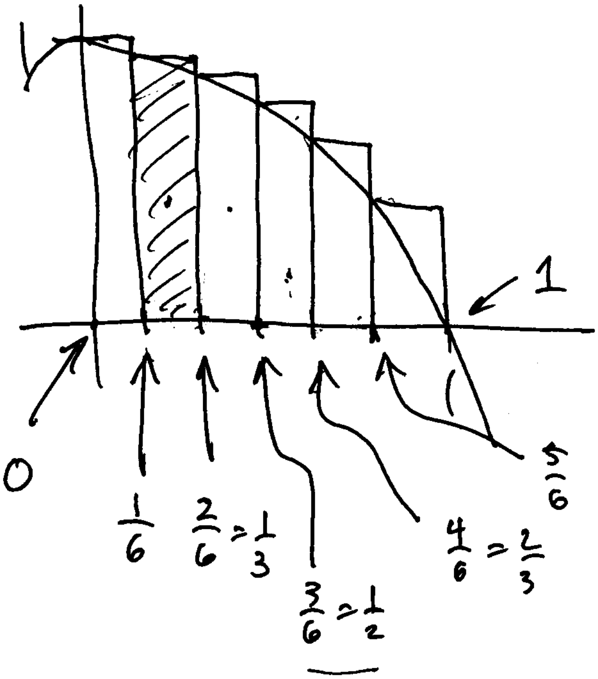


Overestimate of true area.

Upper Sum

Lets look at areas when $n = 6$.

Upper Sum



$$A \approx f(0)\cdot\frac{1}{6} + f\left(\frac{1}{6}\right)\cdot\frac{1}{6} + f\left(\frac{1}{3}\right)\cdot\frac{1}{6} + f\left(\frac{1}{2}\right)\cdot\frac{1}{6} + f\left(\frac{2}{3}\right)\cdot\frac{1}{6} + f\left(\frac{5}{6}\right)\cdot\frac{1}{6}$$

$$= \frac{1}{6}\left[(1-0^2) + \left(1-\left(\tfrac{1}{6}\right)^2\right) + \left(1-\left(\tfrac{1}{3}\right)^2\right) + \left(1-\left(\tfrac{1}{2}\right)^2\right) + \left(1-\left(\tfrac{2}{3}\right)^2\right) + \left(1-\left(\tfrac{5}{6}\right)^2\right)\right]$$

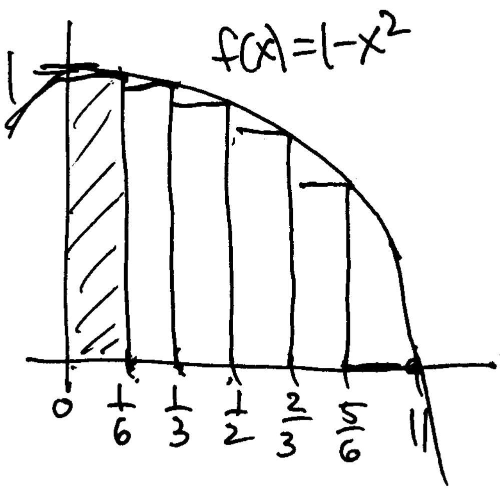
$$= \frac{1}{6}\left[1 + \frac{35}{36} + \frac{8}{9} + \frac{3}{4} + \frac{5}{9} + \frac{11}{36}\right]$$

$$= \frac{1}{6}\left[1 + \frac{46}{36} + \frac{13}{9} + \frac{3}{4}\right]$$

$$= \frac{1}{6}\left[\frac{36+46+52+27}{36}\right]$$

$$= \frac{1}{6}\cdot\frac{161}{36} \approx .74537$$

Lower sum



$$A \approx f\left(\frac{1}{6}\right)\cdot\frac{1}{6} + f\left(\frac{1}{3}\right)\cdot\frac{1}{6} + f\left(\frac{1}{2}\right)\cdot\frac{1}{6} + f\left(\frac{2}{3}\right)\cdot\frac{1}{6} + f\left(\frac{5}{6}\right)\cdot\frac{1}{6} + f(1)\cdot\frac{1}{6}$$

$$= \frac{1}{6}\left[\left(1-\left(\tfrac{1}{6}\right)^2\right) + \left(1-\left(\tfrac{1}{3}\right)^2\right) + \left(1-\left(\tfrac{1}{2}\right)^2\right) + \left(1-\left(\tfrac{2}{3}\right)^2\right) + \left(1-\left(\tfrac{5}{6}\right)^2\right) + \left(1-(1)^2\right)\right]$$

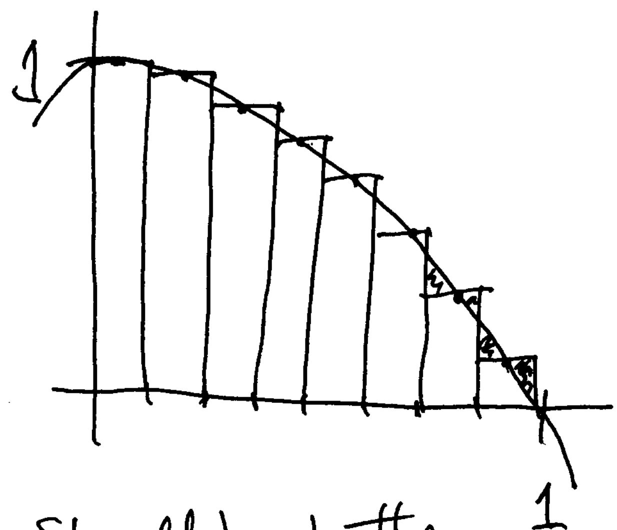
$$= \frac{1}{6}\left[\frac{35}{36} + \frac{8}{9} + \frac{3}{4} + \frac{5}{9} + \frac{11}{36} + 0\right]$$

$$= \frac{1}{6}\left[\frac{46}{36} + \frac{13}{9} + \frac{3}{4}\right] = \frac{1}{6}\left[\frac{46+52+27}{36}\right] = \frac{1}{6}\cdot\frac{125}{36}$$

$$\approx .56944$$

Know: $.56944 \le A \le .74537$

Middle sum: (Do with $n=8$)

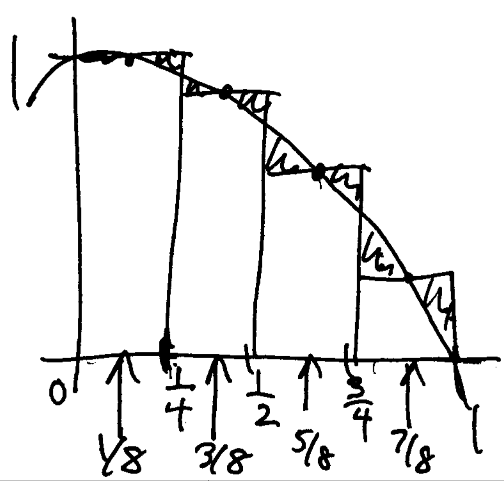


Should be better estimate

On second thought, how about $n=4$.

$$A \approx f\left(\tfrac{1}{8}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{3}{8}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{5}{8}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{7}{8}\right)\cdot\tfrac{1}{4} = \frac{43}{64}$$

$$\approx .67185$$



If we let U_n = upper sum with n subintervals and L_n = lower sum with n subintervals,

i.e. $U_6 = .74537$ $\quad L_6 = .56944$

$U_4 = .78125$ $\quad L_4 = .53125$ (Table 5.1 p 355)

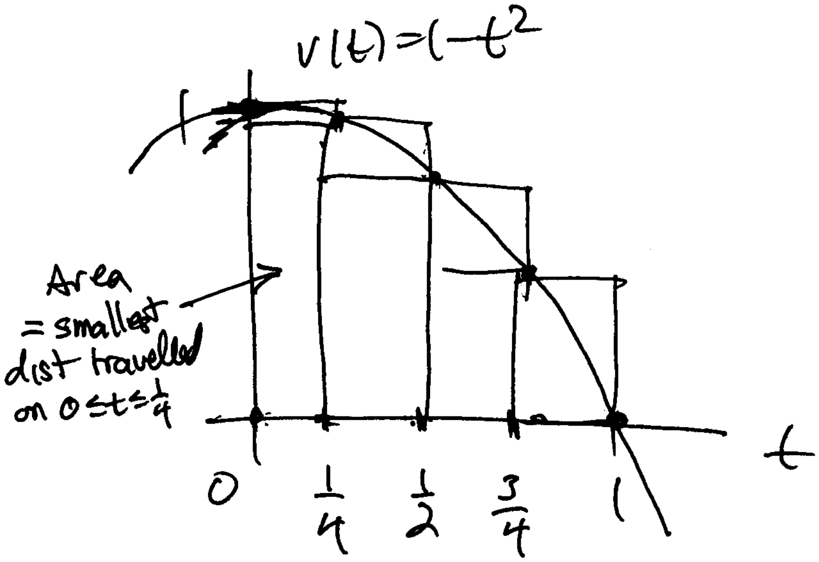
then we note that (1) U_n keeps going down as $n \to \infty$, L_n keeps going up as $n \to \infty$.

(2) U_n and L_n appear to be converging to $2/3$.

Conjecture: $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n = \frac{2}{3}$

so the true area is $\frac{2}{3}$.

Look at related problem: Suppose a particle moves with velocity $v(t) = 1 - t^2$ for $0 \le t \le 1$. How far does it travel?



Consider $0 \le t \le \frac{1}{4}$

We know

$$v(\tfrac{1}{4}) \le v(t) \le v(0)$$

$$\frac{15}{16} \le v(t) \le 1$$

$$\therefore \frac{15}{16} \cdot \frac{1}{4} \le \underline{\text{dist travelled}} \le 1 \cdot \frac{1}{4} = \frac{1}{4} \text{ m}$$

$$\frac{15}{16} \cdot \frac{1}{4} = \frac{15}{64} \text{ m}$$

What about $\frac{1}{4} \le t \le \frac{1}{2}$ $\quad v(\tfrac{1}{2}) \le v(t) \le v(\tfrac{1}{4})$

$$\frac{3}{4} \le v(t) \le \frac{15}{16}$$

$$\frac{3}{4} \cdot \frac{1}{4} \le \underline{\text{dist travelled}} \le \frac{15}{16} \cdot \frac{1}{4}$$

etc...

Finally we estimate total dist travelled by adding up distances over subintervals.

Get:

$$.53125 \le \text{distance travelled} \le .78125$$

Another way: $v(t) = 1 - t^2$ $\quad \underline{s'(t)} = \underline{v(t)}$

$s(t) = $ Anti derivative of $v(t)$

$= t - \frac{1}{3}t^3 + c$

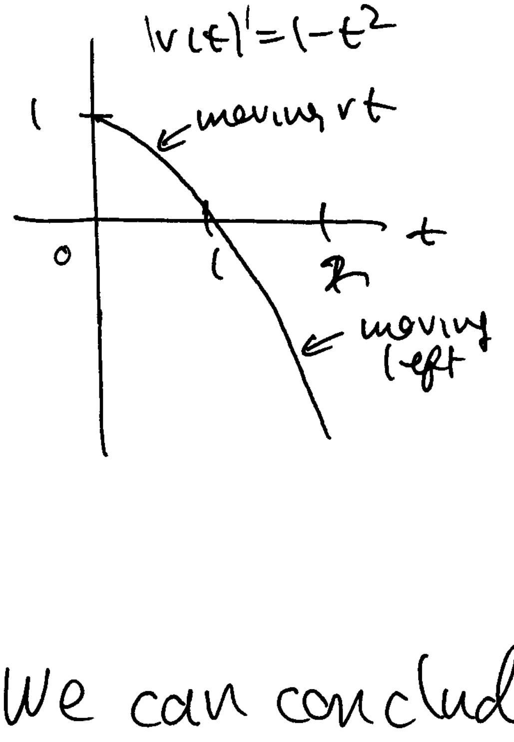
Total dist travelled $=$ total displacement

$= s(1) - s(0) = \left(1 - \frac{1}{3} + c\right) - \left(0 - \frac{1}{3}0^3 + c\right)$

$= \left(\frac{2}{3} + \cancel{c}\right) - (\cancel{c}) = \frac{2}{3}.$

<u>Observations:</u>

① Above only works because $v(t) \geq 0$



Total displacement for $0 \leq t \leq 2$ is

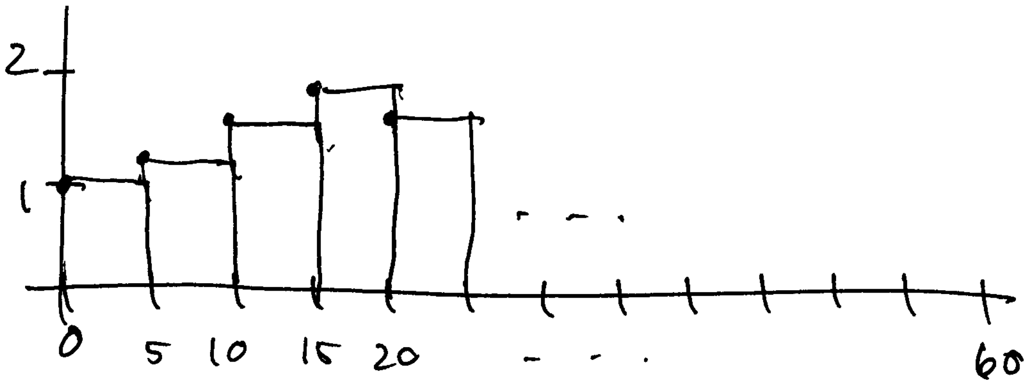
$s(2) - s(0) = \left(2 - \frac{8}{3} + c\right) - (c)$

$= 2 - \frac{8}{3} = -\frac{2}{3}$ m.

<u>Not</u> total dist travelled.

② We can conclude: The total "area" under graph of $f(x)$ (i.e. area above x-axis $-$ area below) $=$ total change in <u>antiderivative of f</u>.

(10)



60

a) Est dist travelled using left endpoints:

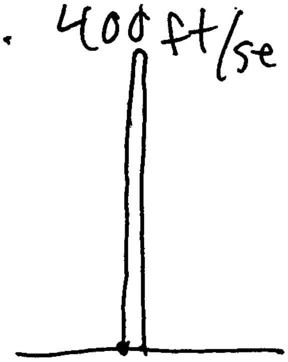
$[0,5]$: dist $= (1 \frac{m}{s})(5 \text{ min})(60 \frac{s}{min}) = 300$ m.

$[5,10]$: dist $= (1.2 \frac{m}{s})(5)(60) = 360$ m

$[10,15]$: dist $= (1.7 \frac{m}{s})(5)(60) = 510$ m

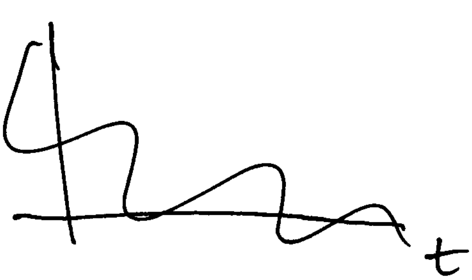
etc ---.

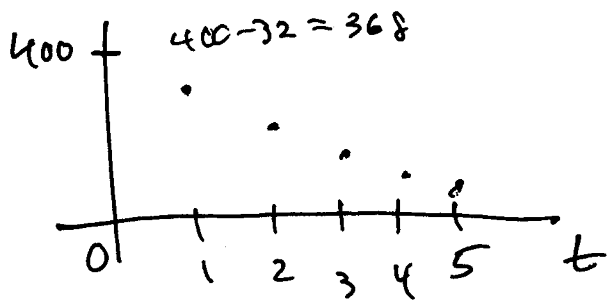
(14) a. 400 ft/sec.



upper estimate for velocity after 5 sec.

Note: velocity is decreasing at rate of 32 ft/sec^2





$a(t) = -32$ $\qquad$ $a(t) = v'(t)$

$v(t) = -32t + C$ $\qquad$ What is C?

$v(0) = 400$

$400 = v(0) = -32(0) + C$

$C = 400$

$v(t) = -32t + 400$

$v(5) = -32(5) + 400 = 240 \text{ ft/sec.}$