4.6 Indeterminate form & L'Hopital's Rule

Indeterminate form

$$\lim_{x\to 1} \frac{x^2-1}{x-1} \qquad \left(\frac{0}{0}\right) \leftarrow \underline{\text{indeterminate form}}$$

$\swarrow$ form of limit

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x\to 1} x+1 = 2$$

eg $$\lim_{x\to 1} \frac{x-1}{x^2+2x-3} \qquad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 1} \frac{x-1}{x^2+2x-3} = \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(x+3)} = \lim_{x\to 1} \frac{1}{x+3} = \frac{1}{4}$$

L'Hopital's Rule: If $f(a) = g(a) = 0$,

$\lim_{x\to a} \frac{f(x)}{g(x)}$ has the indeterminate form $\left(\frac{0}{0}\right)$,

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

e.g. $$\lim_{x\to 1} \frac{x^2-1}{x-1} \overset{L'H}{=} \lim_{x\to 1} \frac{2x}{1} = \lim_{x\to 1} 2x = 2$$

$$\lim_{x\to 1} \frac{x-1}{x^2+2x-3} \overset{L'H}{=} \lim_{x\to 1} \frac{1}{2x+2} = \frac{1}{4}$$

Why? Suppose $\underline{f(a)=0}$ and $\underline{g(a)=0}$

Find linearizations of f and g at $x=a$.

$$\left.\begin{aligned} L_f(x) &= f(a)+f'(a)(x-a) \approx f(x) \\ L_g(x) &= g(a)+g'(a)(x-a) \approx g(x) \end{aligned}\right\} x \text{ near } a.$$

Since $f(a)=0$ and $g(a)=0$

$$L_f(x) = f'(a)(x-a) \approx f(x)$$

$$L_g(x) = g'(a)(x-a) \approx g(x)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)\cancel{(x-a)}}{g'(a)\cdot\cancel{(x-a)}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

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e.g. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-\left(1+\frac{x}{2}\right)}{x^2} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{(1+x)^{1/2}-\left(1+\frac{x}{2}\right)}{x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x}\cdot\frac{(2(1+x)^{1/2})}{(2(1+x)^{1/2})} = \lim_{x\to 0}\frac{1-(1+x)^{1/2}}{4x(1+x)^{1/2}} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0}\frac{-\frac{1}{2}(1+x)^{-1/2}}{\cancel{4}2x\left(\frac{1}{\cancel{2}}(1+x)^{-1/2}\right)+4(1+x)^{1/2}} = \frac{-\frac{1}{2}}{4} = -\frac{1}{8} \;//$$

Other indeterminate forms:

e.g. $\lim_{x\to\infty} \frac{x^2-1}{x^2+2x+3}$ $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+2x+3} = \lim_{x\to\infty} \frac{x^2}{x^2} = \lim_{x\to\infty} 1 = 1.$$

$\lim_{x\to\infty} \frac{x-1}{x^2+2x-3}$ $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{x-1}{x^2+2x-3} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$$

L'Hopital's rule

If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

(works if $a=\infty$ or if a is finite) then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Idea: look at $\lim_{x\to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$. This has the

indet form $\left(\frac{0}{0}\right)$ and by L'H,

$$\lim_{x\to a} \frac{g(x)^{-1}}{f(x)^{-1}} = \lim_{x\to a} \frac{-g(x)^{-2}\cdot g'(x)}{-f(x)^{-2}\cdot f'(x)} = \left(\lim_{x\to a} \frac{f(x)}{g(x)}\right)^2 \cdot \lim_{x\to a} \frac{g'(x)}{f'(x)}$$

$$\cancel{\lim_{x\to a}\frac{f(x)}{g(x)}} = \left(\lim_{x\to a}\frac{f(x)}{g(x)}\right)^{\cancel{2}} \cdot \lim_{x\to a}\frac{g'(x)}{f'(x)}$$

End up with $\displaystyle \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

e.g. $\displaystyle \lim_{x\to\infty}\frac{x^2-1}{x^2+2x+3} \overset{L'H}{=} \lim_{x\to\infty}\frac{2x}{2x+2} \overset{L'H}{=} \lim_{x\to\infty}\frac{2}{2} = 1$

(still $\frac{\infty}{\infty}$)

e.g. $\displaystyle \lim_{x\to\infty}\frac{x-1}{x^2+2x-3} \overset{L'H}{=} \lim_{x\to\infty}\frac{1}{2x+2} = 0$

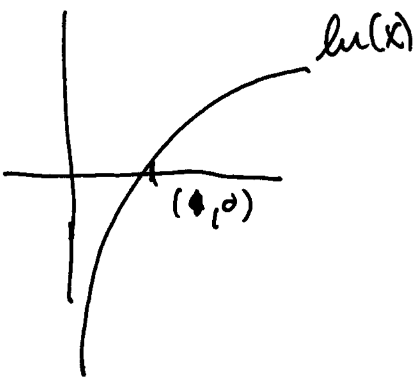
Another indet form:

$0\cdot\infty$

e.g. $\displaystyle \lim_{x\to 0^+} x\ln(x)$ has the form $(0\cdot\infty)$

$(0\cdot -\infty)$

How to do this? Put it in the form $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ + apply L'H.



$\left(\frac{0}{0}\right)$: $\displaystyle \lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+}\frac{x}{\left(\frac{1}{\ln(x)}\right)}$

OR

$\left(\frac{\infty}{\infty}\right)$: $\displaystyle \lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0}\frac{\ln(x)}{\left(\frac{1}{x}\right)}$

L'H works on one and not on the other.

(g): $\lim_{x\to 0^+} \frac{x}{\left(\frac{1}{\ln x}\right)} = \lim_{x\to 0^+} \frac{x}{(\ln x)^{-1}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{1}{-(\ln(x))^{-2}\cdot\frac{1}{x}}$

$= \lim_{x\to 0^+} \frac{x}{-(\ln x)^{-2}} \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 0^+} \frac{1}{2(\ln x)^{-3}\cdot\frac{1}{x}}$

$= \lim_{x\to 0^+} \frac{x}{2(\ln x)^{-3}} \left(\frac{0}{0}\right)$ L'H fails

$\left(\frac{\infty}{\infty}\right)$: $\lim_{x\to 0^+} \frac{\ln(x)}{\left(\frac{1}{x}\right)} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}(-x^2)$

$\left[\, x^{-1} \to -x^{-2} = -\frac{1}{x^2} \,\right]$ $\quad = \lim_{x\to 0^+} -x = 0$ //

More indeterminate forms:

$0^0, \ 1^\infty, \ \infty^0$

e.g. $\lim_{x\to 0^+} x^x$. How to do this?

$\ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x) = 0$

$\therefore \lim_{x\to 0^+} x^x = e^{\ln\left(\lim_{x\to 0^+} x^x\right)} = e^0 = 1$

e.g. $\lim_{x\to 1^+} x^{\left(\frac{1}{x-1}\right)}$ $\quad (1^\infty)$

$$\ln\left(\lim_{x\to 1^+} x^{\left(\frac{1}{x-1}\right)}\right) = \lim_{x\to 1^+} \ln\left(x^{\frac{1}{x-1}}\right)$$

$$= \lim_{x\to 1^+} \frac{1}{x-1}\ln(x) = \lim_{x\to 1^+} \frac{\ln x}{x-1} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{1} = \lim_{x\to 1^+} \frac{1}{x} = 1$$

$$\therefore \lim_{x\to 1^+} x^{\frac{1}{x-1}} = e^{\ln\left(\lim_{x\to 1^+} x^{\frac{1}{x-1}}\right)} = e^1 = e$$

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eg $\lim_{x\to\infty} x^2 e^{-x}$ $\quad (\infty\cdot 0)$

$$= \lim_{x\to\infty} \frac{x^2}{e^x} \quad \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

In fact $\lim_{x\to\infty} x^n e^{-x} = 0$ for any $n > 0$.

Exponential decay beats polynomial growth.

$$\lim_{x\to 0^+} x^{1/2} \ln(x) \quad (0\cdot\infty)$$

$$= \lim_{x\to 0^+} \frac{x^{1/2}}{(\ln x)^{-1}}$$ won't work.

$$= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} \frac{1}{x}\cdot(-2)x^{3/2} = \lim_{x\to 0^+} -2x^{1/2} = 0$$

Fact: $\lim_{x\to 0^+} x^{\alpha} \ln(x) = 0$ for any $\alpha > 0$.

eg $\lim_{x\to 0^+} x^{1/10} \ln(x) = 0$

e.g. $\lim_{x\to\infty} \frac{\ln(x)}{x} \quad \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

$$\lim_{x\to\infty} \frac{\ln(x)}{x^{1/5}} \quad \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{5}x^{-4/5}} = \lim_{x\to\infty} 5x^{4/5}\cdot\frac{1}{x}$$

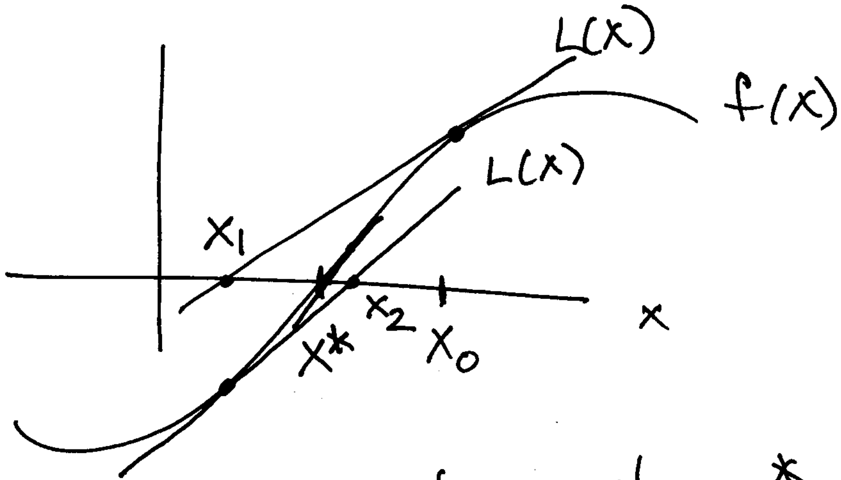
$$= \lim_{x\to\infty} \frac{5}{x^{1/5}} = 0.$$

Fact: $\lim_{x\to\infty} \frac{\ln(x)}{x^{\alpha}} = 0$ for any $\alpha > 0$

4.7 Newton's Method.

Idea: Want to solve equation $f(x)=0$ where f is some given function. We call a solution to $f(x)=0$ a root of $f(x)$.
Sometimes can't be done by algebra, so need to do it numerically.

Method:



① First make a guess at root x^*, call it x_0.

② Find linearization $L(x)$ of $f(x)$ at $x=x_0$

③ Then solve $L(x)=0$. This solution should be closer to x^*

④ Call this solution x_1 and repeat.

Formula: x_0 is a guess.

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$0 = f(x_0) + f'(x_0)(x - x_0) \qquad \text{solve for } x$$

$$f'(x_0)(x - x_0) = -f(x_0)$$

$$x f'(x_0) - x_0 f'(x_0) = -f(x_0)$$

$$x f'(x_0) = x_0 f'(x_0) - f(x_0)$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \; x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

e.g. Find root of $f(x) = x^2 - 3$

$x^2 - 3 = 0$

$x = \sqrt{3} \quad x = -\sqrt{3}$

$(\sqrt{3} = 1.732050808\ldots)$

Newton iteration scheme:

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n}}$$

Make a guess for x_0. Say $x_0 = 1.5$

$x_0 = 1.5$

$$x_1 = 1.5 - \frac{(1.5)^2 - 3}{2(1.5)} = 1.5 - \frac{2.25 - 3}{3} = 1.75$$

$$x_2 = 1.75 - \frac{(1.75)^2 - 3}{2(1.75)} = 1.732142857\ldots$$

$x_3 = 1.732050810$

e.g. Find a root of $f(x) = x^6 - x - 1$

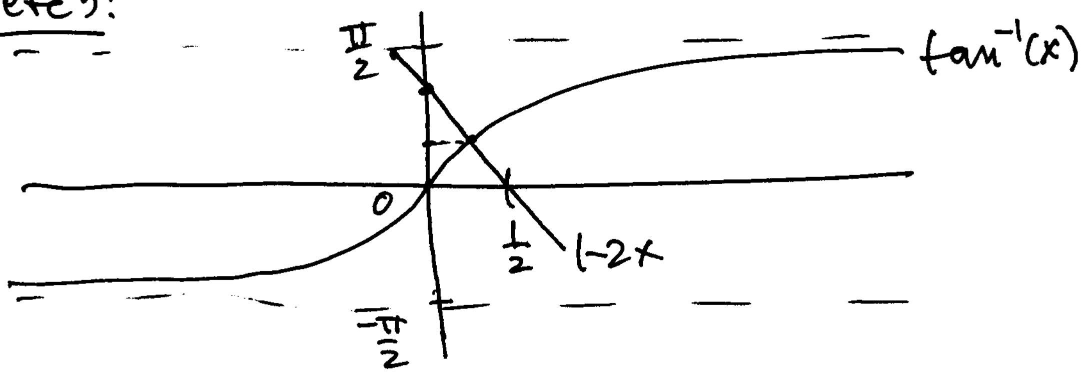
Newton iteration: $f'(x) = 6x^5 - 1$

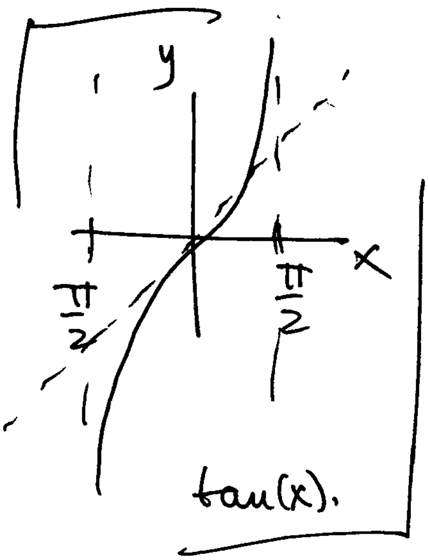
$$\boxed{x_{n+1} = x_n - \frac{x_n^6 - x_n - 1}{6x_n^5 - 1}}$$

Pick x_0. Say $x_0 = 1$

e.g. #6 $\tan^{-1}(x) = 1-2x$

Sketch:





How to do Newton?

$$f(x) = \tan^{-1}(x) - 1 + 2x$$

$f(x) = 0$ means $\tan^{-1}(x) - 1 + 2x = 0$

or $\tan^{-1}(x) = 1 - 2x$

Could also use:

$$g(x) = \frac{\tan^{-1}(x)}{1-2x} - 1$$

$$\tan^{-1}(x) = 1-2x$$

$$\frac{\tan^{-1}(x)}{1-2x} = 1$$

$$\frac{\tan^{-1}(x)}{1-2x} - 1 = 0$$

$$f'(x) = \frac{1}{1+x^2} + 2$$

$$\boxed{x_{n+1} = x_n - \frac{\tan^{-1}(x_n) - 1 + 2x_n}{\frac{1}{1+x_n^2} + 2}}$$

$x_0 = .25$ $\qquad$ $x_0 = 0$