Exam 3 3.7–4.4

Final Exam 6-19 (Tuesday)
$9^{\underline{30}} - 1^{\underline{20}}$ (Dont panic)

Review 3.8 (17) 3.7 51) 41) 3.9 7) 13)
4.1 29)

3.7 41) $y = \sqrt{x(x+1)}$

$$\ln(y) = \ln([x(x+1)]^{1/2}) = \frac{1}{2}\ln(x(x+1))$$
$$= \frac{1}{2}(\ln(x) + \ln(x+1))$$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left(\frac{1}{2}(\ln(x) + \ln(x+1))\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}y\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{1}{2}\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

51) $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

$$\ln(y) = \ln\left(\frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}}\right)$$

$$= \ln(x(x^2+1)^{1/2}) - \ln((x+1)^{2/3})$$

$$= \ln(x) + \ln((x^2+1)^{1/2}) - \ln((x+1)^{2/3})$$

$$= \ln(x) + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1)$$

Take $\frac{d}{dx}$ of both sides.

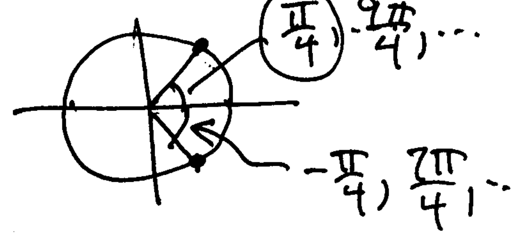
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+1} - \frac{2}{3}\,\frac{1}{x+1}$$

$$\frac{dy}{dx} = y\cdot\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right)$$

$$= \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right)$$ //

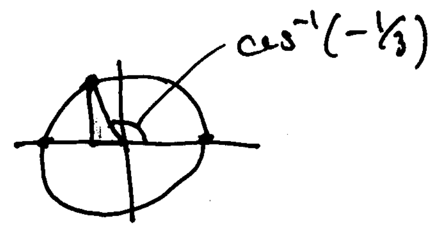
3.8 17)

$$\sin\Big(\underbrace{\cos^{-1}\left(\tfrac{\sqrt{2}}{2}\right)}_{\frac{\pi}{4}}\Big)$$



$$= \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

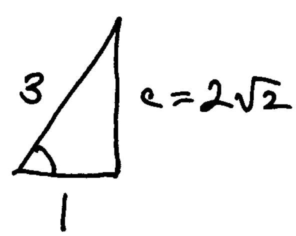
<u>eg.</u> $\sin\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)$



$$\sin\left(\cos^{-1}\left(-\tfrac{1}{3}\right)\right) = \frac{2\sqrt{2}}{3}$$

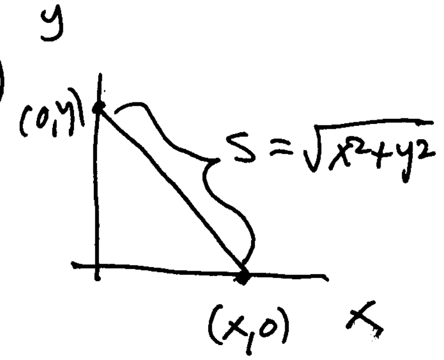
$$\tan\left(\cos^{-1}\left(-\tfrac{1}{3}\right)\right) = -2\sqrt{2}$$

etc...



$$1^2 + c^2 = 3^2 = 9$$
$$c^2 = 8$$
$$c = \sqrt{8} = 2\sqrt{2}$$

3.9 7)



$$S = (x^2+y^2)^{1/2}$$

$$\frac{d}{dt}(S) = \frac{d}{dt}\left((x^2+y^2)^{1/2}\right)$$

$$\frac{dS}{dt} = \frac{1}{2}(x^2+y^2)^{-1/2}\frac{d}{dt}(x^2+y^2)$$

$$= \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}\left(\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}\right)$$

$$= (x^2+y^2)^{-1/2}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

(a) If y constant then $\frac{dy}{dt} = 0$

$$\frac{dS}{dt} = x(x^2+y^2)^{-1/2}\frac{dx}{dt}.$$

(b) If nothing is constant then

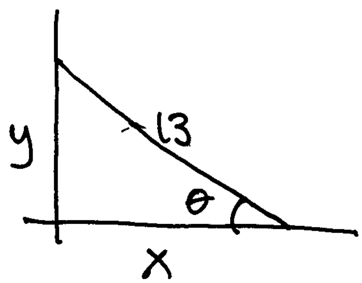
$$\frac{dS}{dt} = (x^2+y^2)^{-1/2}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

(c) If S constant then $\frac{dS}{dt} = 0$

$$0 = (x^2+y^2)^{-1/2}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \qquad \frac{1}{(x^2+y^2)^{1/2}} \neq 0$$

Simplifying: $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ //

(13)



(a) Want $\frac{dy}{dt}$. Given x, $\frac{dx}{dt}$

$$x^2+y^2=13^2=169$$

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(169)$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$(12)(5)+(5)\left(\frac{dy}{dt}\right)=0 \longleftarrow x\frac{dx}{dt}+y\frac{dy}{dt}=0$$

$$\boxed{\frac{dy}{dt}=-12\ \text{ft/sec}}$$

Need y: $\underline{x=12\text{ ft}}$ $\quad \frac{dx}{dt}=5\ \text{ft/sec}$

$$(12)^2+y^2=(13)^2$$

$$144+y^2=169$$

$$y^2=169-144=25$$

$$\underline{y=5\text{ ft}}$$

(b) $A=\frac{1}{2}xy$

$$\frac{d}{dt}(A)=\frac{d}{dt}\left(\frac{1}{2}xy\right)$$

$$\frac{dA}{dt}=\frac{1}{2}\left(x\frac{dy}{dt}+y\frac{dx}{dt}\right)$$

$$\frac{dA}{dt}=\frac{1}{2}\left(12\cdot(-12)+(5)(5)\right)$$

$$=\frac{1}{2}(25-144)$$

$$=-\frac{119}{2}\ \text{ft}^2/\text{sec}$$

(c) $\sin(\theta)=\frac{y}{13}$

$$\frac{d}{dt}(\sin(\theta))=\frac{d}{dt}\left(\frac{y}{13}\right)$$

$$\underbrace{\cos(\theta)}\frac{d\theta}{dt}=\frac{1}{13}\frac{dy}{dt}$$

$\hookrightarrow$ $\cos(\theta)=\frac{x}{13}=\frac{12}{13}$

$x=12$

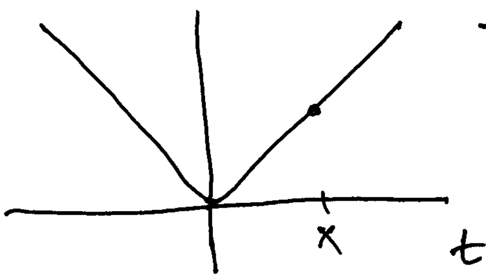
$$\frac{12}{13}\cdot\frac{d\theta}{dt} = \frac{1}{13}(-12)$$

$$\frac{12}{13}\,\frac{d\theta}{dt} = -\frac{12}{13}$$

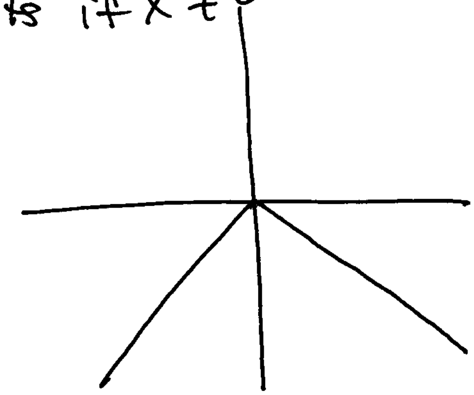
$$\boxed{\frac{d\theta}{dt} = -1 \text{ rad/sec.}}$$

4.1 29) $f(t) = 2-|t| \quad -1 \le t \le 3$

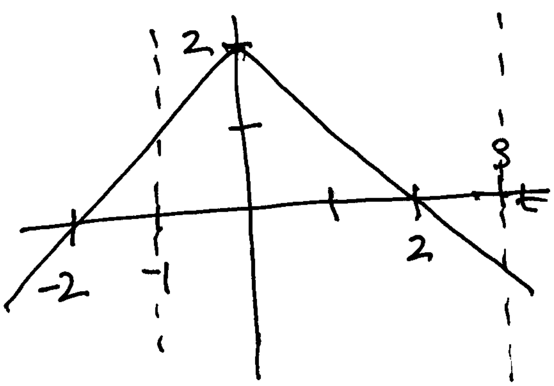
$f'(x)$ exists if $x \neq 0$



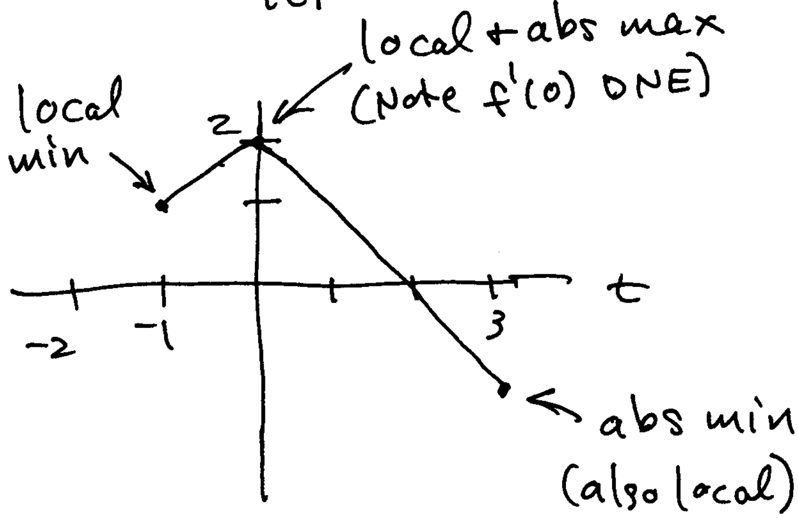
$|t|$



$-|t|$



$2-|t|$



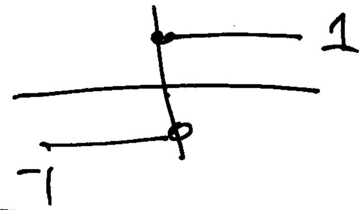
$$f(t) = \begin{cases} 2-t & t \ge 0 \\ 2+t & t < 0 \end{cases}$$

$$|t| = \sqrt{t^2} = (t^2)^{1/2}$$

$$\frac{d}{dt}|t| = \frac{1}{2}(t^2)^{-1/2}(\cancel{2}t)$$

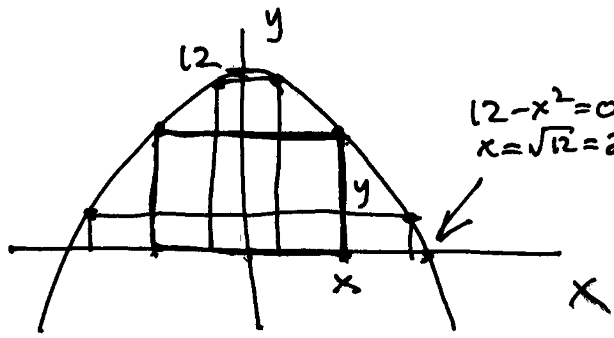
$$f'(t) = \begin{cases} -1 & t > 0 \\ 1 & t < 0 \\ \text{UND.} & t = 0 \end{cases}$$

$$= \frac{t}{\sqrt{t^2}} = \frac{t}{|t|}$$



4.5 Applied Optimization Problems.

#4, p309



$12 - x^2 = 0$
$x = \sqrt{12} = 2\sqrt{3}$

$y = 12 - x^2$

$0 \le x \le 2\sqrt{3}$

Maximize area of rectangle.

$$A = 2xy = 2x(12 - x^2) = 24x - 2x^3$$

Find crit pts of A.

$$A'(x) = 24 - 6x^2$$

$$24 - 6x^2 = 0$$

$$6x^2 = 24$$

$$x^2 = 4$$

$$x = 2$$

Check:

$A(2) = 48 - 16 = 32$

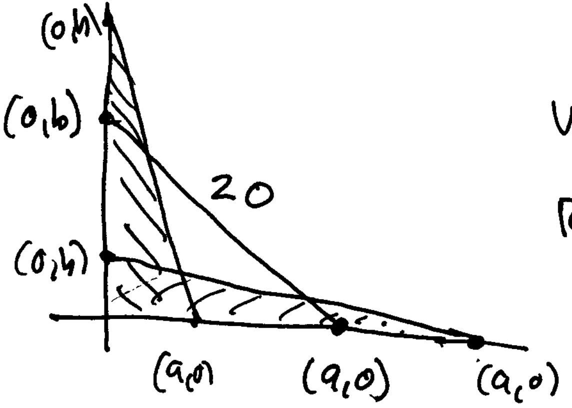
$A(0) = 0$

$A(2\sqrt{3}) = 0$

Largest area $= 32$

Dimensions: $\boxed{\begin{array}{l} x = 2 \\ y = 8 \end{array}}$

so rect is 4×8 units //

#6) 

Variables: a, b.

Relationship: $a^2+b^2=400$

Maximize: $A=\frac{1}{2}ba$

Range of a:
$0 \le a \le 20$

$$A=\frac{1}{2}ba = \frac{1}{2}a(400-a^2)^{1/2}$$

$$a^2+b^2=400$$
$$b^2=400-a^2$$
$$b=(400-a^2)^{1/2}$$

Find max value of A if $0 \le a \le 20$

Critical points:

$$A' = \frac{1}{2}\left(a \cdot \frac{1}{\cancel{2}}(400-a^2)^{-1/2}(-\cancel{2}a) + (400-a^2)^{1/2}\right)$$

$$= \frac{1}{2}\left(-a^2(400-a^2)^{-1/2} + (400-a^2)^{1/2}\right)$$

$$= \frac{1}{2}\left(\frac{-a^2}{(400-a^2)^{1/2}} + (400-a^2)^{1/2}\right)$$

$$= \frac{1}{2}\left(\frac{-a^2+400-a^2}{(400-a^2)^{1/2}}\right)$$

$$= \frac{1}{2}\left(\frac{400-2a^2}{(400-a^2)^{1/2}}\right) = \frac{200-a^2}{(400-a^2)^{1/2}}$$

Crit pts: $a=(200)^{1/2}$, $a=20$
$A'=0$ A' undef.

$A(\sqrt{200}) = \frac{1}{2}\sqrt{200}\,(400-200)^{1/2} = 100$ //

$A(20) = 0$

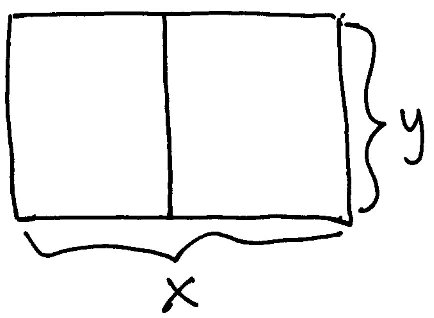
$A(0) = 0$

Max area is 100 when $a = \sqrt{200}$,

What is b?

$b = (400-200)^{1/2} = \sqrt{200}$,

#8)



$xy = 216$

Minimize length of fence

$L = 2x + 3y$

$L = 2x + 3\left(\frac{216}{x}\right)$ $y = \frac{216}{x}$

$L = 2x + \frac{648}{x}$

Range of x: $0 < x < \infty$

$x = ~~\ ~~\ 216$
$y = 1$
$L = 435$
is this min? No
$x = 128$
$y = 2$
$L = 256 + 6 = 262$

Crit pts.

$L'(x) = 2 - \frac{648}{x^2}$

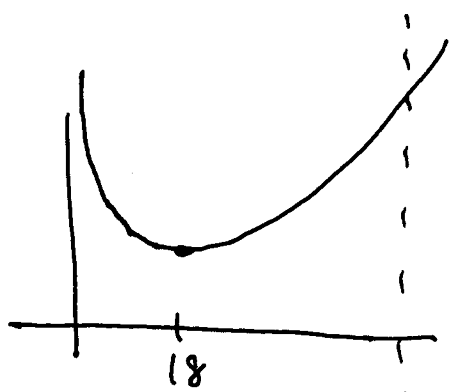
$2 - \frac{648}{x^2} = 0$

$\frac{648}{x^2} = 2$

$648 = 2x^2$

$x^2 = 324$

$x = 18$ ✓

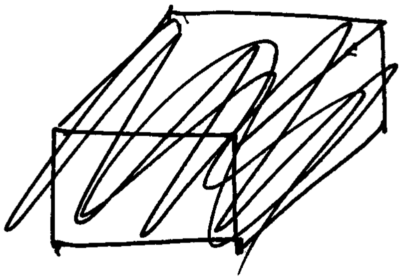


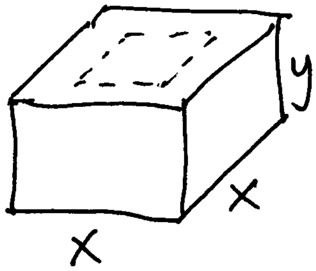
Minimum length at $x = 18\,m$

$y = 12\,m = \frac{216}{18}$

Min length of $L = 2\cdot 18 + 3\cdot 12 = 72\,m$

#10)





$$x^2 y = 1125$$

Minimize $\quad C = \underbrace{5(x^2+4xy)}_{\text{cost of materials}} + \underbrace{10xy}_{\text{excavation charge}}$

Think about C:
Amount of material $\approx$ Area of sides + bottom.

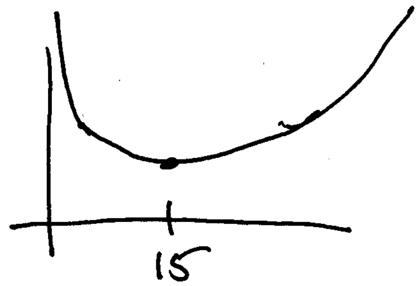
$$\underset{\text{bottom}}{x^2} + \underset{\text{sides}}{4xy}$$

$$C = 5(x^2+4xy) + 10xy \qquad x^2y = 1125$$

$$= 5x^2 + 30xy \qquad y = \frac{1125}{x^2}$$

$$C = 5x^2 + 30x\left(\frac{1125}{x^2}\right)$$

$$= 5x^2 + \frac{33750}{x}$$



Critical points:

$$C'(x) = 10x - \frac{33750}{x^2}$$

$$10x = \frac{33750}{x^2}$$

$$10x^3 = 33750$$

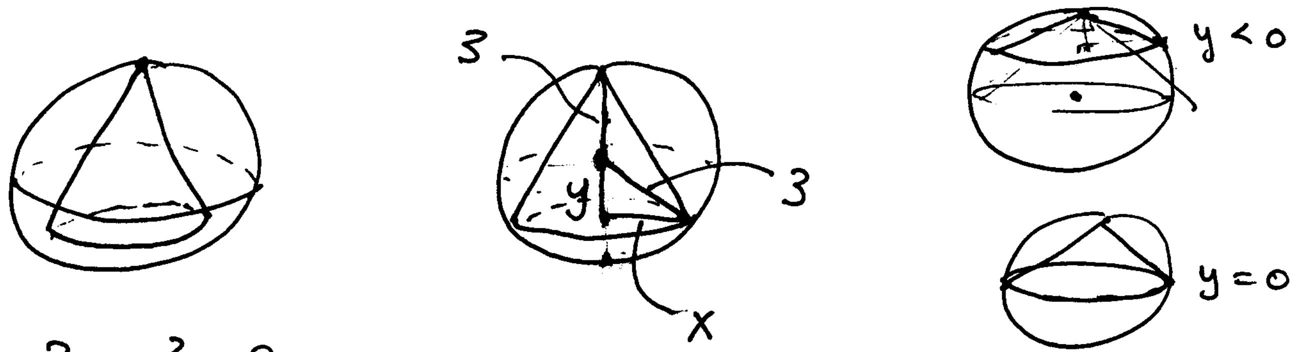
$$x^3 = 3375$$

$$x = 15$$

$$y = \frac{1125}{(15)^2} = \frac{1125}{225} = 5$$

Cost minimized when $x = 15$, $y = 5$ //

#12)



$$x^2 + y^2 = 9$$

Maximize Volume of cone, V.

$$V = \frac{1}{3}\pi x^2 (y+3) \qquad x^2 + y^2 = 9$$

$$x^2 = 9 - y^2$$

$$V = \frac{1}{3}\pi (9 - y^2)(y+3)$$

Range of y:

$$-3 \le y \le 3$$

$$V = \frac{1}{3}\pi(-y^3 - 3y^2 + 9y + 27)$$

Crit pts: $V'(y) = \frac{1}{3}\pi(-3y^2 - 6y + 9)$

$$= \frac{1}{\cancel{3}}\pi(-\cancel{3})(y^2 + 2y - 3)$$

$$= -\pi(y-1)(y+3)$$

Crit pts: $y = 1$, $y = -3$

$V(1) = \frac{1}{3}\pi(8)(4) = \frac{32\pi}{3}$ // $V(-3) = 0$

$V(3) = 0$

Max volume is $\frac{32\pi}{3}$ when $y = 1$ and $x = \sqrt{8}$