MAPLE #2 due tomorrow

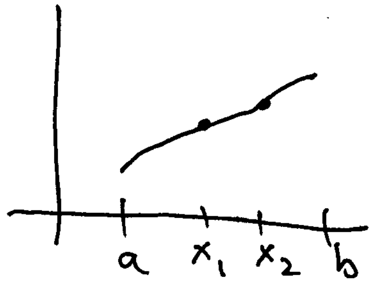
Exam 3 3.7 – 4.4 (Friday)

MAPLE #3 will be available on web.

Local extrema always occur at critical points (and at endpoints of closed interval), that is, $x=c$ is a critical point if: (a) $f'(c)=0$ or (b) $f'(c)$ is undefined.

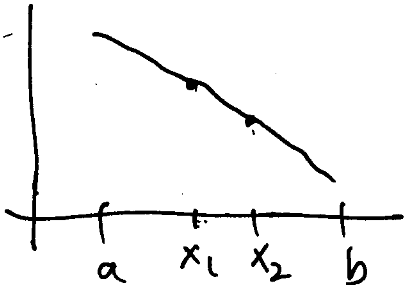
4.3 Monotonic functions and 1st derivative test

Idea: If $f'(x)>0$ at each x in (a,b) then $f(x)$ is increasing on $[a,b]$, i.e., if $x_1<x_2$ then $f(x_1)<f(x_2)$.



increasing.

If $f'(x)<0$ at each x in (a,b) then $f(x)$ is decreasing on $[a,b]$, i.e. if $x_1<x_2$ then $f(x_1)>f(x_2)$.



e.g. #10) p289 $g(t) = -3t^2 + 9t + 5$

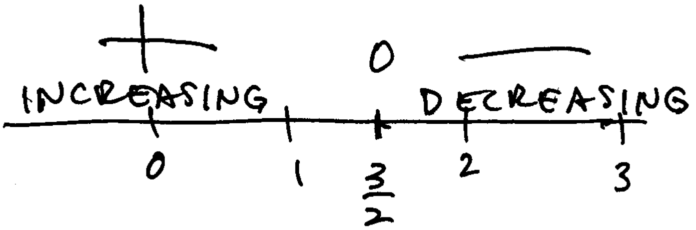
(1) Find where g is increasing / decreasing.

$g'(t) = -6t + 9$

Find critical points. $-6t + 9 = 0$

$6t = 9$

$\boxed{t = \frac{3}{2}}$ // ← critical point.



$g'(t)$

$g'(2) = -3 < 0$

$g'(0) = 9 > 0$

g(t) decreasing on $(3/2, \infty)$

g(t) increasing on $(-\infty, 3/2)$

(2) Find all local extrema + extreme values.

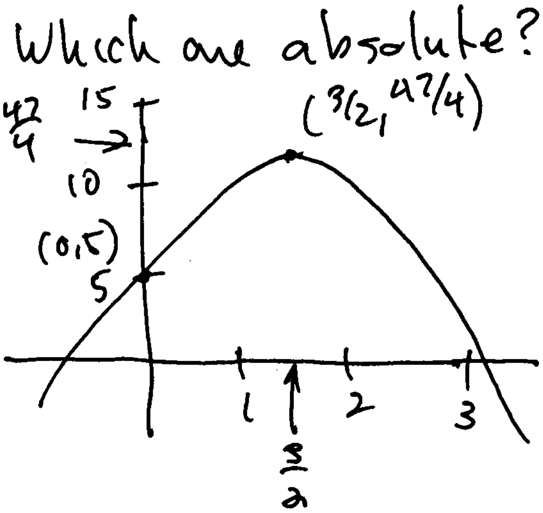
Local max at $t = 3/2$ with value

$g(3/2) = -3\left(\frac{3}{2}\right)^2 + 9\left(\frac{3}{2}\right) + 5$

$= -\frac{27}{4} + \frac{27}{2} + 5$

$= \frac{27}{4} + \frac{20}{4} = \frac{47}{4}$ //

(3) Which one absolute?



Absolute max at $t = 3/2$ with abs max value $47/4$.

OR Absolute max at the point $(3/2, 47/4)$.

eg #(8) $g(x) = x^4 - 4x^3 + 4x^2$

① Find where $g(x)$ is increasing/decreasing

$g'(x) = 4x^3 - 12x^2 + 8x$

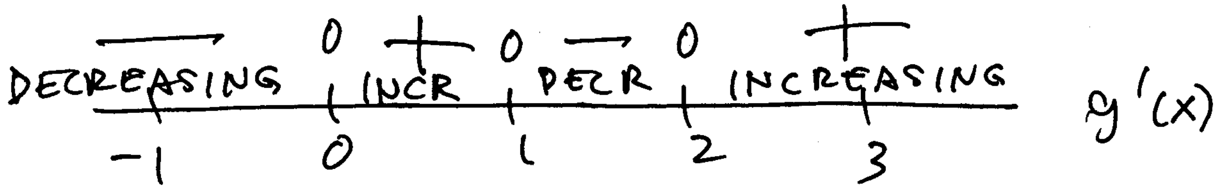
Find critical points

$4x^3 - 12x^2 + 8x = 0$

$4x(x^2 - 3x + 2) = 0$

$4x(x-1)(x-2) = 0$

$\underline{x = 0, \ x = 1, \ x = 2 \quad \text{crit. pts}}$



$g'(-1) < 0$

$g'(\frac{1}{2}) > 0$

$g'(\frac{3}{2}) < 0$

$g'(3) > 0$

$g'(x) = 4x(x-1)(x-2)$

(−)	(−)	(−)
(+)	(−)	(−)
(+)	(+)	(−)
(+)	(+)	(+)

$g(x)$ is increasing on $(0,1) \cup (2, \infty)$

" " decreasing " $(-\infty, 0) \cup (1,2)$ //

(2) Find all local extrema + extreme values

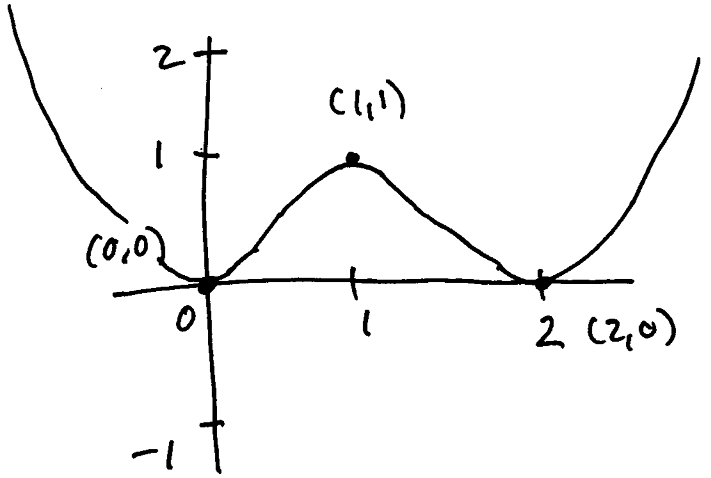
Local min at x = 0 of $g(0) = 0$

Local max at x = 1 of $g(1) = 1$

Local min at x = 2 of $g(2) = 16 - 32 + 16 = 0$

(3) Which of these are absolute?

3 points: $(0,0)$ $(1,1)$ $(2,0)$



Absolute minima ~~at~~ of 0 at $x=0$ and $x=2$.

e.g. #20 $K(t) = 15t^3 - t^5$

(1) $K'(t) = 45t^2 - 5t^4$

$k'(-4) < 0$

$k'(-1) > 0$

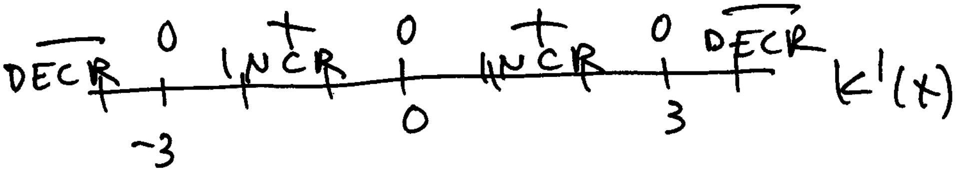
$k'(1) > 0$

$k'(4) < 0$

$45t^2 - 5t^4 = 0$

$5t^2(9 - t^2) = 0$

$t = 0$, $t = -3$, $t = 3$



$K(t)$ increasing on $(-3, 3)$ [OK to say $(-3,0) \cup (0,3)$].

K(t) decreasing on $(-\infty,-3)\cup(3,\infty)$

(2) Local extrema

Local min at $x=-3$ of $K(-3)=15(-3)^3-(-3)^5$

$$= -27\cdot 15-(-243)$$

$$= -162$$

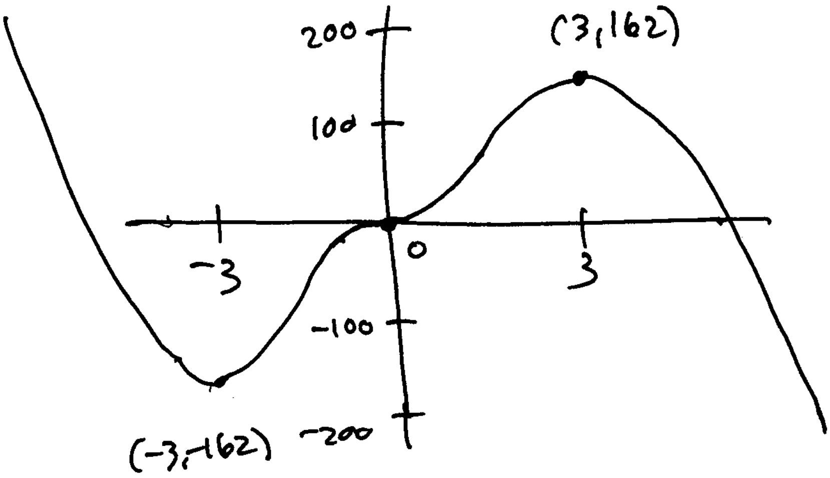
$x=0$ is neither a local max nor local min.

Local max at $x=3$ of $K(3)=162$

(3) Which are absolute?

(0,0) (3,162)

(-3,-162)



None are absolute extrema.

eg #28) $k(x)=x^{2/3}(x^2-4)=x^{8/3}-4x^{2/3}$

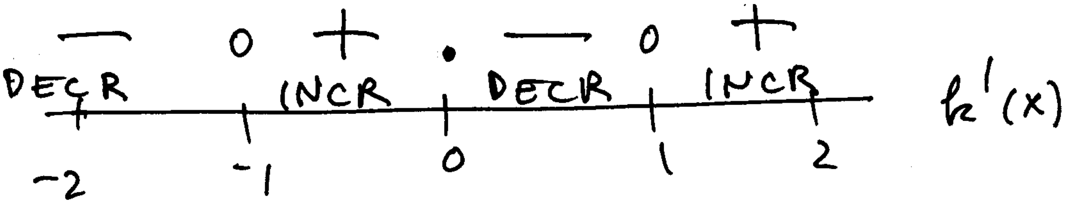
(1) $k'(x)=\frac{8}{3}x^{5/3}-\frac{8}{3}x^{-1/3}=\frac{8}{3}\left(x^{5/3}-x^{-1/3}\right)$

$$=\frac{8}{3}\left(x^{5/3}-\frac{1}{x^{1/3}}\right)=\frac{8}{3}\left(\frac{x^2-1}{x^{1/3}}\right)$$

$$\boxed{x^{5/3}=\frac{x^{5/3}\cdot x^{1/3}}{x^{1/3}}=\frac{x^2}{x^{1/3}}}$$

Critical points: $\underbrace{x=1\,,\,x=-1}_{k'=0}$

$\underbrace{x=0}_{k' \text{ undefined}}$



$k'(-2) < 0$, $k'(-1/2) > 0$, $k'(1/2) < 0$, $k'(2) > 0$

$k(x)$ is increasing on $(-1,0) \cup (1,\infty)$

decreasing $(-\infty,-2) \cup (0,1)$

(2) Local extrema.

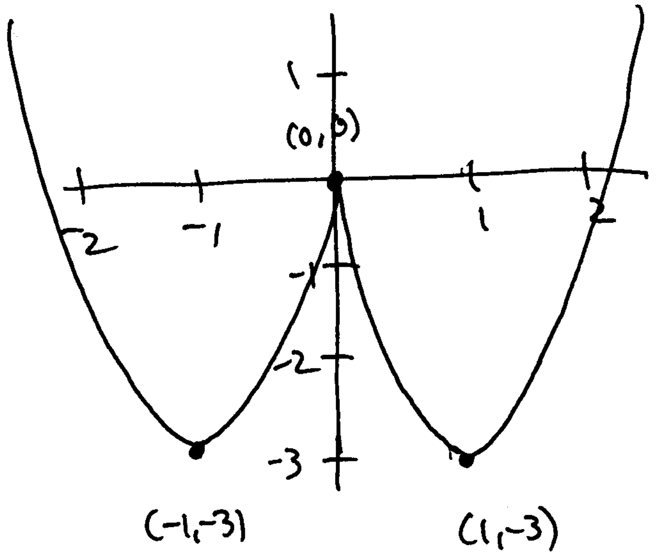
Local min at $x=-1$ of $k(-1) = (-1)^{8/3} - 4(-1)^{2/3}$

$= 1 - 4 = -3$

Local max at $x=0$ of $k(0) = 0$

Local min at $x=1$ of $k(1) = -3$

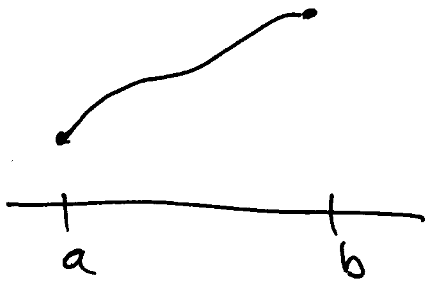
(3) Which are absolute? $(-1,-3)$ $(0,0)$ $(1,-3)$



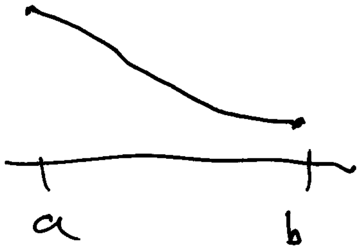
Absolute minima at $x = 1, -1$ of -3.

4.4 Concavity and curve sketching

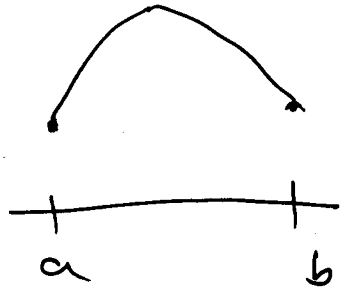
Question: We know that $f'(x)$ tells us where $f(x)$ is increasing and decreasing. What does $f''(x)$ tell us?



monotonically increasing on $[a,b]$



monotonically decreasing on $[a,b]$

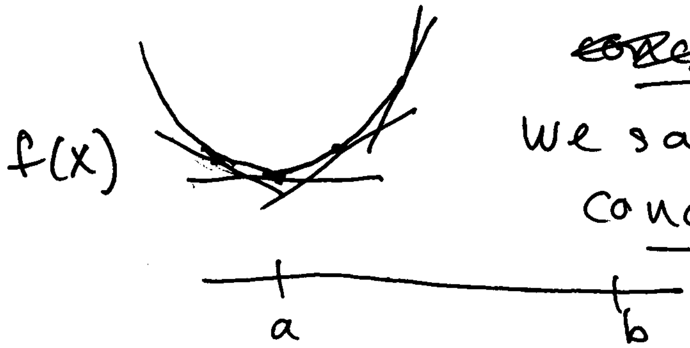


not monotonic

would say "f(x) is monotonic"

Idea: (1) $f''(x)$ is first derivative of $f'(x)$ so $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing.

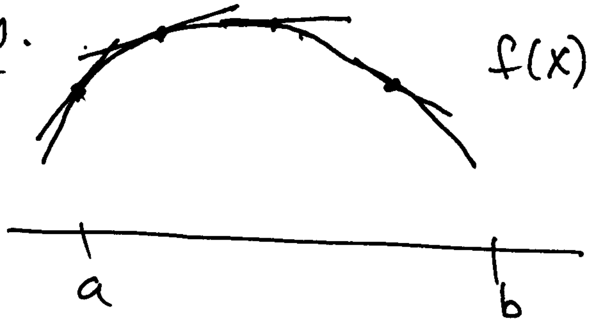
(2) Suppose $f''(x)>0$ on (a,b). Then $f'(x)$ is increasing on $[a,b]$.



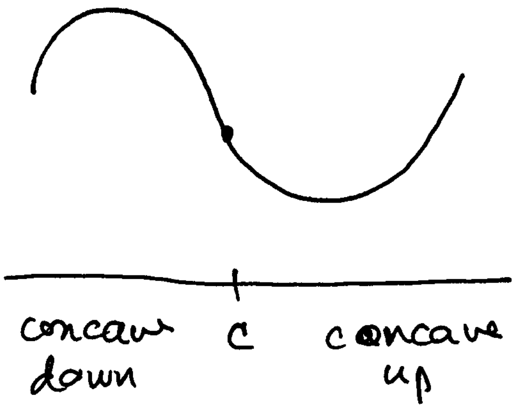
~~concave up~~

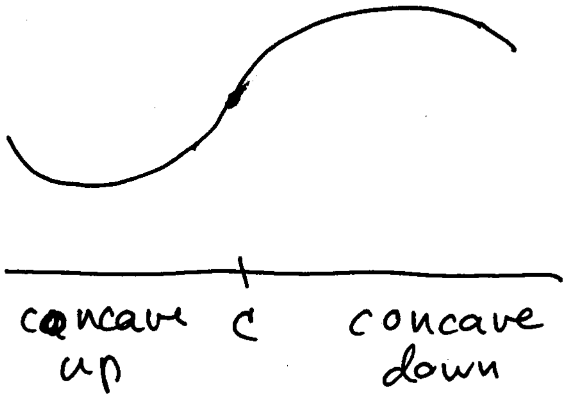
We say $f(x)$ is concave up

(3) Suppose $f''(x)<0$ on (a,b). Then $f'(x)$ is decreasing on $[a,b]$. We say $f(x)$ is concave down.



~~when~~ If $f''(x)$ changes sign (from positive to negative or negative to positive) at $x=c$ then we say $x=c$ is an inflection point





e.g. #10 p289

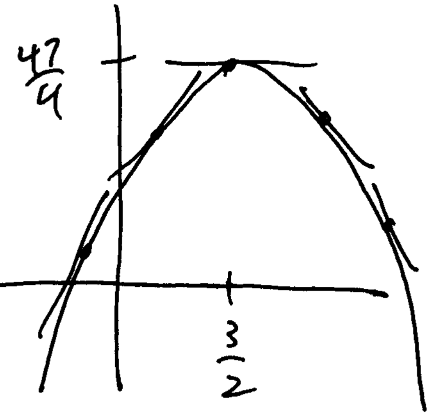
$g(t) = -3t^2 + 9t + 5$

$g'(t) = -6t + 9$ crit point $t = \frac{3}{2}$

INCR $\overset{0}{\underset{\frac{3}{2}}{|}}$ DECR g'

$g(3/2) = \frac{47}{4}$

$g''(t) = -6$ Since $g''(t) < 0$ always, $g(t)$ is always concave down.



e.g. $g(x) = x^4 - 4x^3 + 4x^2$

$g'(x) = 4x^3 - 12x^2 + 8x$

crit pts $x = 0, 1, 2$

$x = 0, 2$: local min

$x = 1$: local max

$g''(x) = 12x^2 - 24x + 8$

Find intervals where $g(x)$ is concave up/down.

$12x^2 - 24x + 8 = 0$

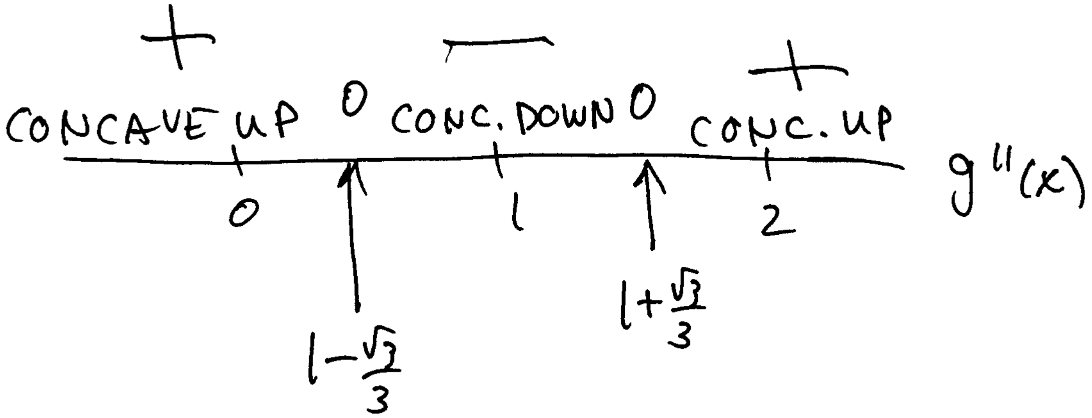
$4(3x^2 - 6x + 2) = 0$

$x = \frac{6 \pm \sqrt{36-24}}{6}$

$$x = \frac{6 \pm \sqrt{12}}{6} = \frac{\overset{3}{\cancel{6}} \pm \cancel{2}\sqrt{3}}{\cancel{6}3} = \frac{3 \pm \sqrt{3}}{3} = 1 \pm \frac{\sqrt{3}}{3}$$

possible inflection points

$$\approx \begin{cases} 1.58 \approx 1 + \frac{\sqrt{3}}{3} \\ .42 \approx 1 - \frac{\sqrt{3}}{3} \end{cases}$$

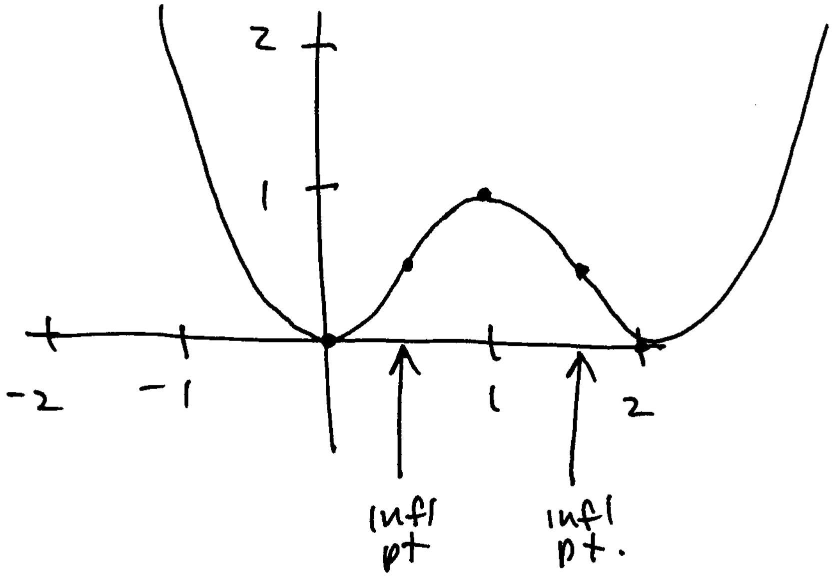


$g''(0) = 8 > 0$, $g''(1) = 12-24+8 = -4 < 0$, $g''(2) = 48-48+8 > 0$

INFLECTION POINTS AT $x = 1 - \frac{\sqrt{3}}{3}$ and

$x = 1 + \frac{\sqrt{3}}{3}$. //

Graph.



eg $K(t) = 15t^3 - t^5$

$K'(t) = 45t^2 - 5t^4$

crit pts: $t = 0, -3, 3$

neither, local min, local max

Look at concavity.

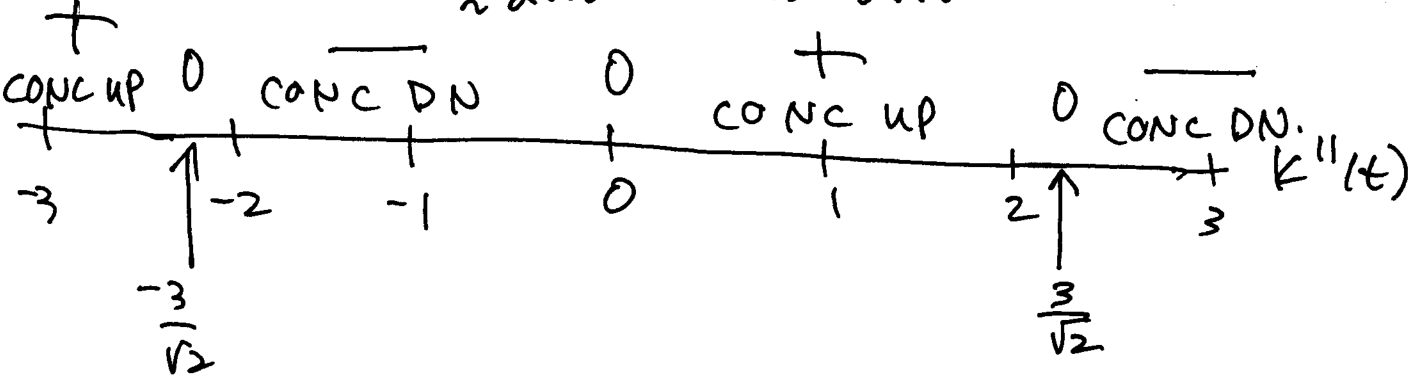
$K''(t) = 90t - 20t^3$

$90t - 20t^3 = 0$

~~$10t(9 - 2t^2) = 0$~~

$20t\left(\frac{9}{2} - t^2\right) = 0$

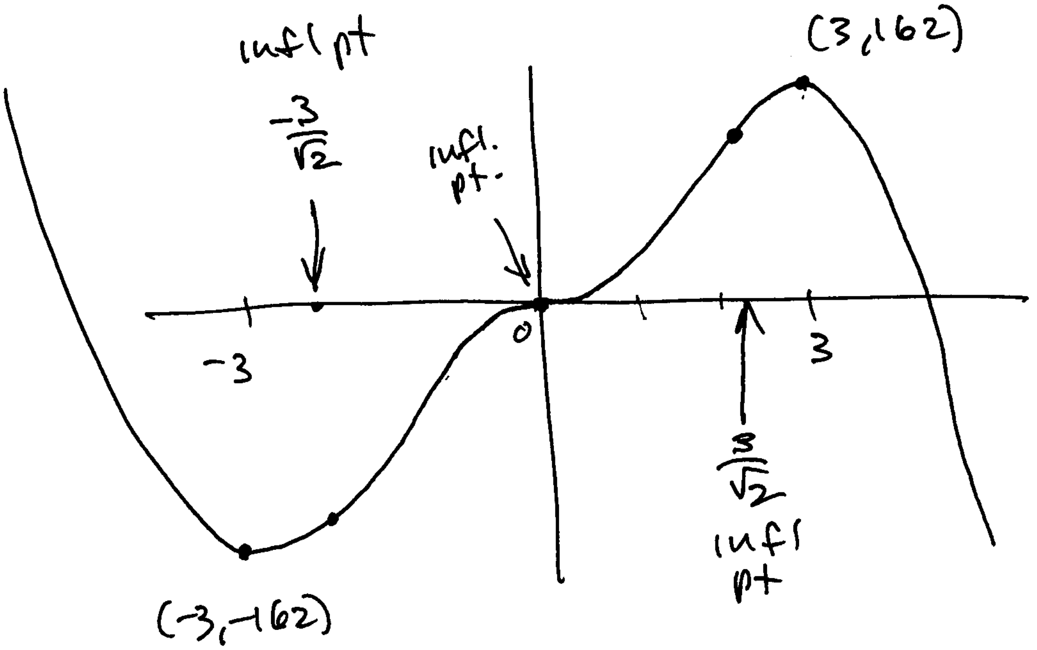
$t = 0$, $t = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} \approx 2.12$, $t = -\sqrt{\frac{9}{2}} = -\frac{3}{\sqrt{2}} \approx -2.12$ } possible inflection points.



$K''(-3) > 0$, $K''(-1) < 0$, $K''(1) > 0$, $K''(3) < 0$

Inflection points at $x = \frac{-3}{\sqrt{2}}, 0, \frac{3}{\sqrt{2}}$.

Sketch: $(-3,-162)$ $(0,0)$ $(3,162)$



e.g. $k(x) = x^{2/3}(x^2-4)$

$$k'(x) = \frac{8}{3}\left(\frac{x^2-1}{x^{1/3}}\right) = \frac{8}{3}\left(x^{5/3} - x^{-1/3}\right)$$

$$\text{CP: } \underbrace{x=1, x=-1}_{k'=0}, \underbrace{x=0}_{k' \text{ und.}}$$

loc min at $x=-1, 1$

loc. max at $x=0$.

Look at concavity.

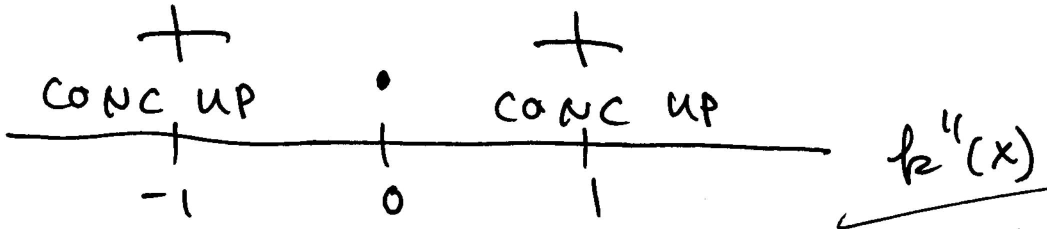
$$k''(x) = \frac{8}{3}\left(\frac{5}{3}x^{2/3} + \frac{1}{3}x^{-4/3}\right)$$

$$= \frac{8}{9}\left(5x^{2/3} + x^{-4/3}\right)$$

$$= \frac{8}{9}\left(5x^{2/3} + \frac{1}{x^{4/3}}\right)$$

$$= \frac{8}{9}\left(\frac{5x^2+1}{x^{4/3}}\right)$$

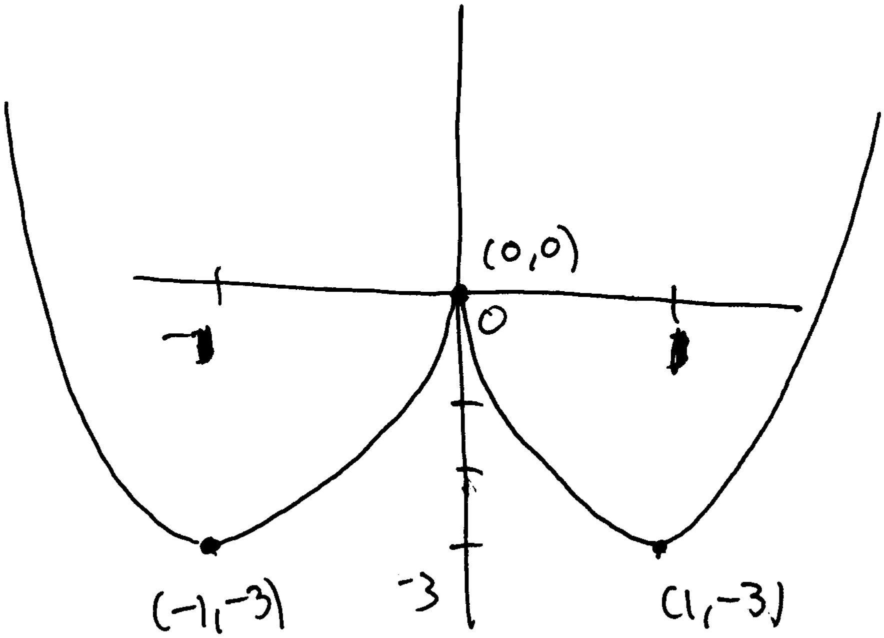
possible inflection pt: $x=0$ (k'' undef)



$k''(-1) > 0$ $k''(1) > 0$

$\boxed{\text{No inflection points}}$

$(-1,-3)$ $(0,0)$ $(1,-3)$



concave up everywhere (except <ins>at</ins> $x=0$ where k'' undefined)