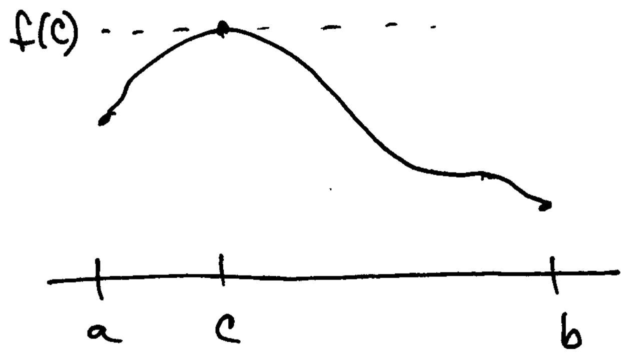
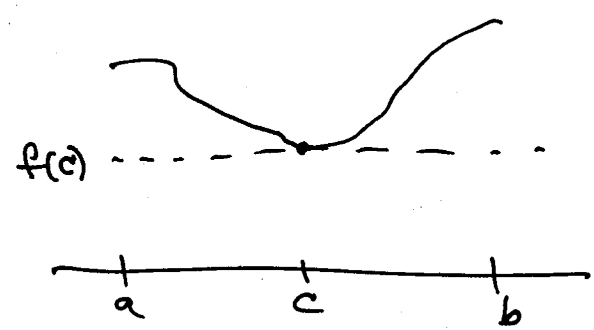
Maple #2 due Thursday
Exam 3 Friday

<u>4.1</u> Extreme values

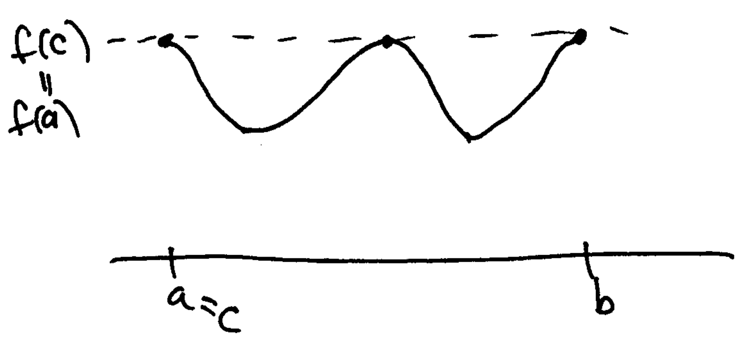
<u>Def</u>: A function $f(x)$ on an interval I (could be open / closed / finite / infinite) has an <u>absolute</u> <u>maximum</u> at $x=c$ if $f(x) \leq f(c)$ for all x in I, and an <u>absolute</u> <u>minimum</u> at $x=c$ if $f(x) \geq f(c)$ for all x in I.



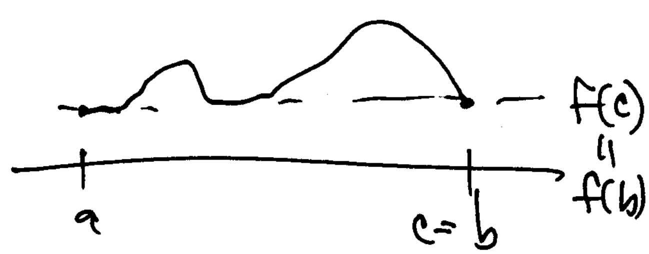
absolute maximum



absolute minimum

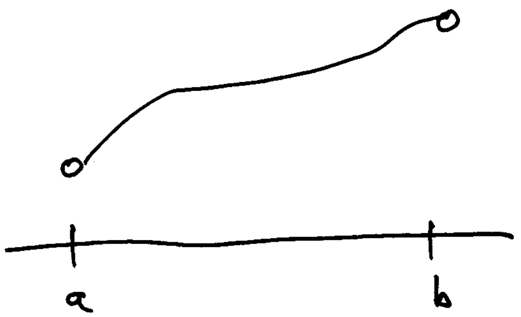


abs max
attained at endpoints



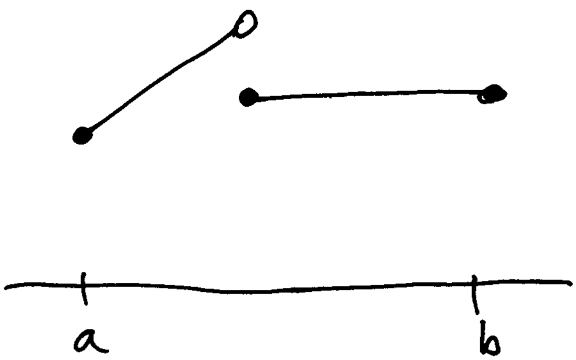
abs min.
attained at endpoint

Q: When does a function attain its absolute max or min (we say "absolute extrema")?

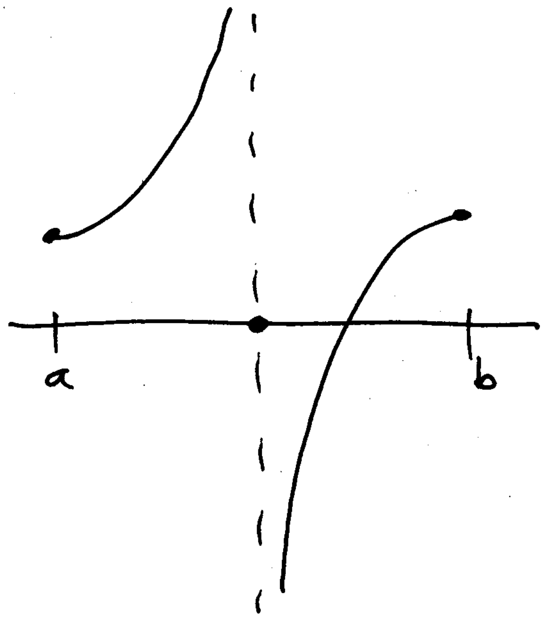


Here $I = (a,b)$ and $f(x)$ does not attain its max or min.

$I = [a, \infty)$ $f(x)$ does not attain maximum but does attain min.

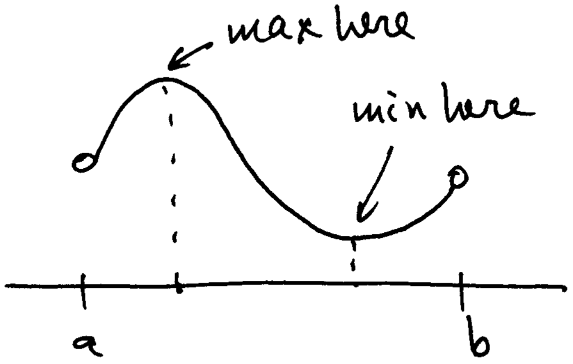


Here $I = [a,b]$. Does not attain max.

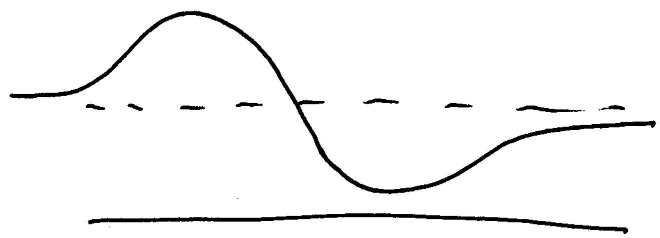


$I = [a,b]$. Does not ~~att~~ attain max or min.

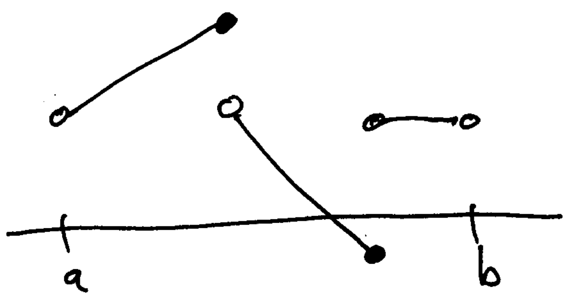
Thm: If $f(x)$ is continuous on $I=[a,b]$ (closed, finite interval), then f attains its max and min in I.



$I=(a,b)$, attains max + min

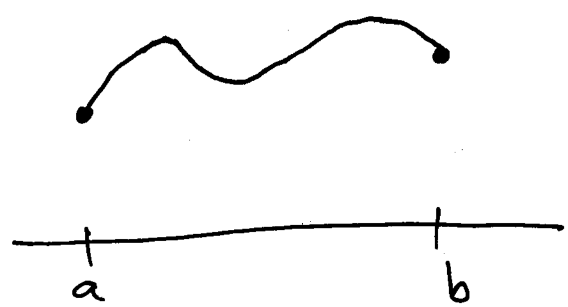


$I=(-\infty,\infty)$, attains both max + min.

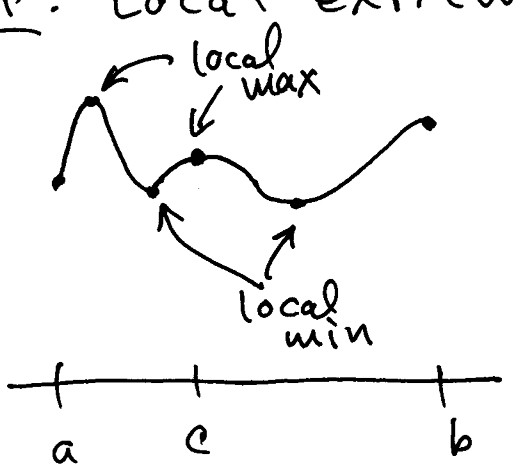


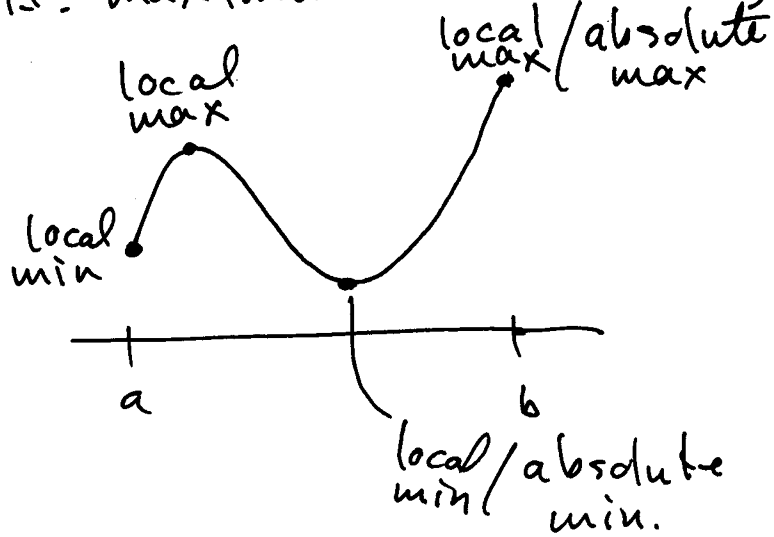
$I=(a,b)$, f discont, attains max/min.

Why is Thm true?

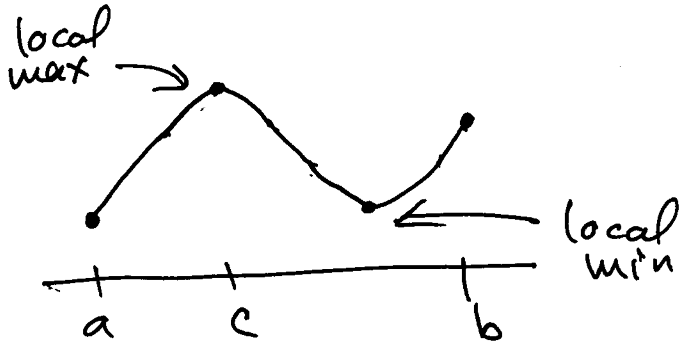


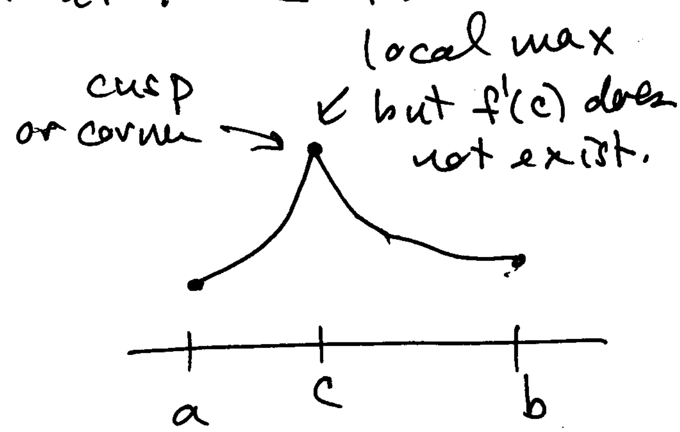
Def: Local extrema (i.e. maximums + minimums)





Thm: If $f(x)$ is differentiable at $x=c$ in the interior of I, and f has a local max or min at $x=c$ then $f'(c)=0$.





This means that the local maximums or minimums of a continuous function on $I=[a,b]$ occur either at:

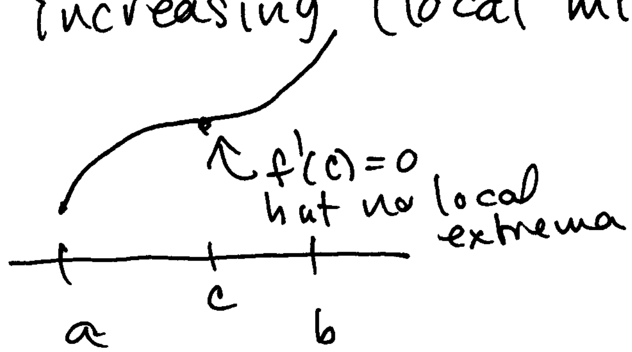
① endpoints.

② a point c in (a,b) where $f'(c)=0$

③ a point c in (a,b) where $f'(c)$ does not exist

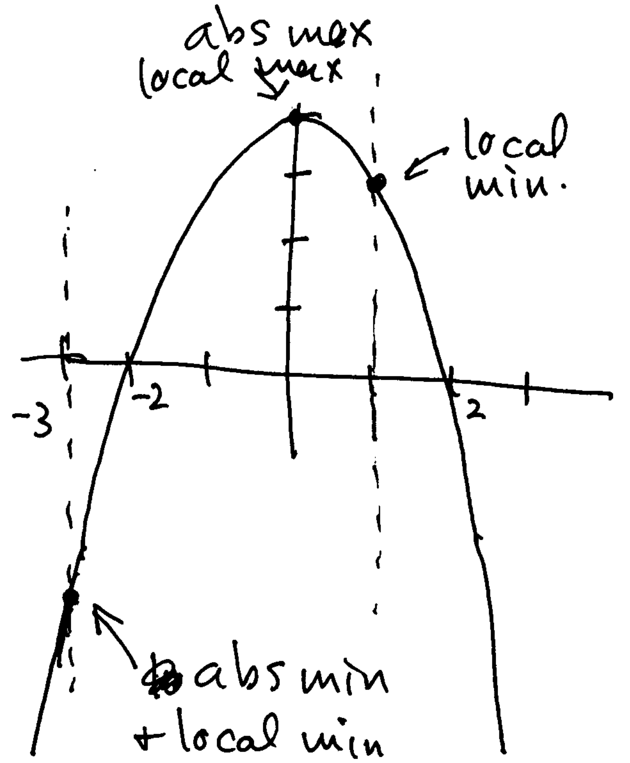
Def: A critical point of $f(x)$ is a point where either $f'=0$ or f' does not exist.

Fact: $f(x)$ can change from increasing to decreasing (local max) or decreasing to increasing (local min) at a critical point.



Fact: Absolute extrema of a function $f(x)$ must occur either at the endpoints of I or at critical points.

eg #18 $f(x) = 4 - x^2$ $\quad -3 \le x \le 1 \quad I = [-3, 1]$



Find critical points.

$$f'(x) = -2x$$

$$f'(x) = 0 : \quad -2x = 0$$

$$\underline{x = 0}$$

Check: endpts $f(-3) = -5$ abs. min

$f(1) = 3$

crit pt. — $f(0) = 4$ abs max

eg #24) $g(x) = -(5 - x^2)^{1/2}$ $\quad -\sqrt{5} \le x \le \sqrt{5}$

crit pts

$$g'(x) = +\tfrac{1}{\cancel{2}}(5 - x^2)^{-1/2}(+\cancel{2}x)$$

$$= x(5 - x^2)^{-1/2}$$

$$= \frac{x}{(5 - x^2)^{1/2}}$$

$g'(x) = 0$

$$\frac{x}{(5 - x^2)^{1/2}} = 0 \quad \underline{x = 0}$$

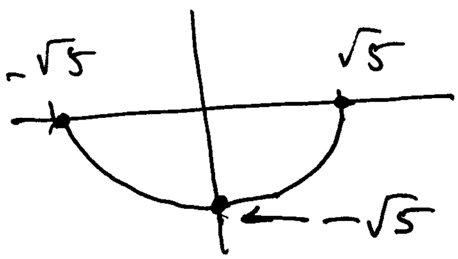
$g'(x)$ undefined $\quad \underline{x = \pm\sqrt{5}}$

Check:

$g(-\sqrt{5}) = 0$ abs max

$g(\sqrt{5}) = 0$

$g(0) = -\sqrt{5}$ abs min



e.g #42 $y = x^3 - 3x^2 + 3x - 2$

Find all extreme values

Find critical points.

NO EXTREME POINTS

$$y' = 3x^2 - 6x + 3$$

$$3x^2 - 6x + 3$$

$$3(x^2 - 2x + 1) = 0$$

$$3(x-1)^2 = 0$$

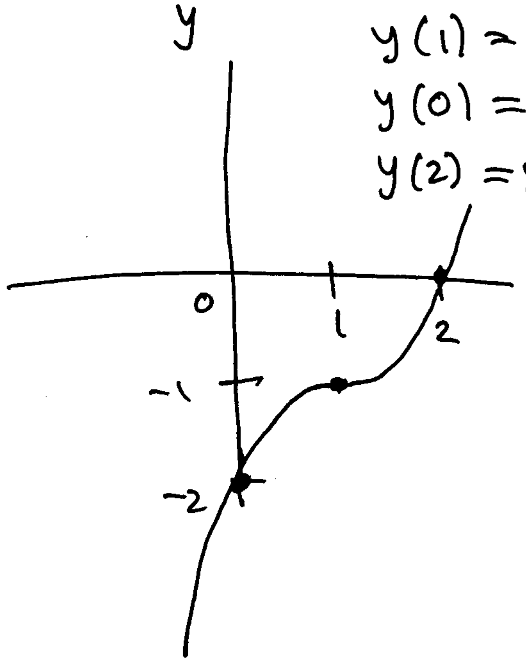
$$x = 1$$

What is happening at $x=1$?

$y(1) = -1$

$y(0) = -2$

$y(2) = 8 - 12 + 6 - 2 = 0$



eg #43

$$y = \sqrt{x^2 - 1}$$

Defined on $(-\infty, -1] \cup [1, \infty)$

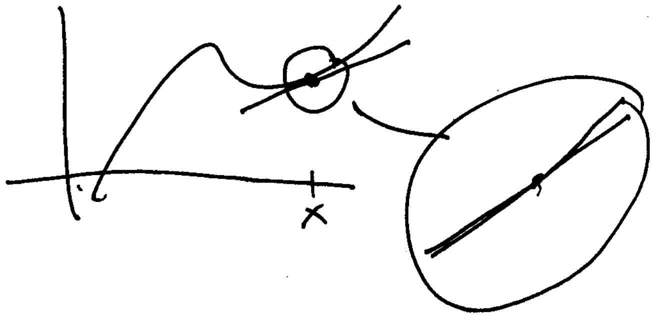
Need to check $y(-1)$ and $y(1)$ for extreme values.

4.2 Mean Value Theorem

<u>Ideas</u>: (1) Relationship between average rates of change and instantaneous rates of change.

(2) Formalize the idea that a function "follows" its tangent lines.

(3) The only functions whose derivatives are zero everywhere are constant functions.

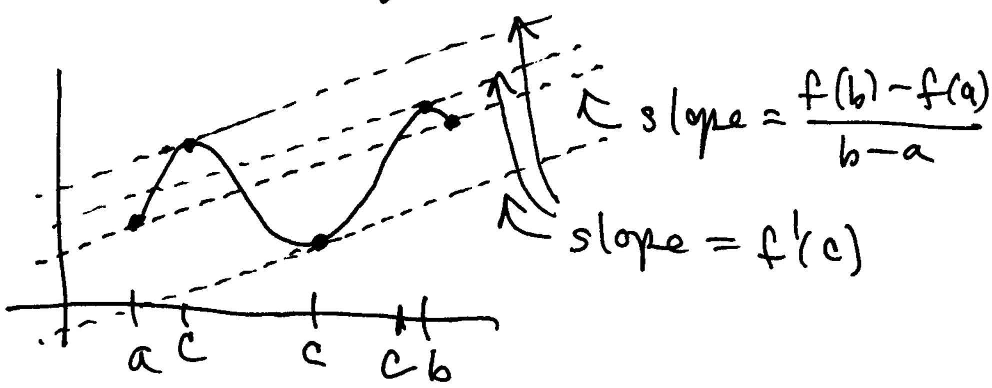


(1) <u>Mean Value Theorem</u>:

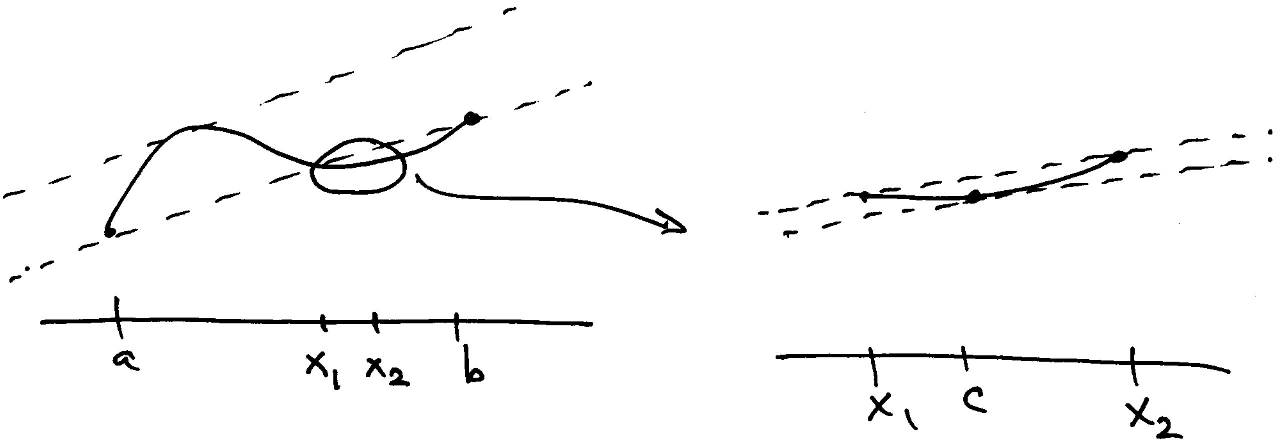
If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there is a point c in (a,b) such that

$$\underbrace{\frac{f(b)-f(a)}{b-a}}_{\substack{\text{avg rate of change}\\ \text{of } f \text{ on } [a,b].\\ \text{(Difference quotient)}}} = \underbrace{f'(c)}_{\substack{\text{instantaneous}\\ \text{rate of change at } c.}}$$

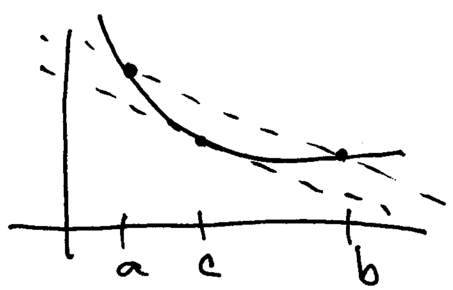
~~instantaneous~~



(2)



For example: If I know that $f'(x) > 0$ at every point x in (a,b) then I know that $f(b) > f(a)$.

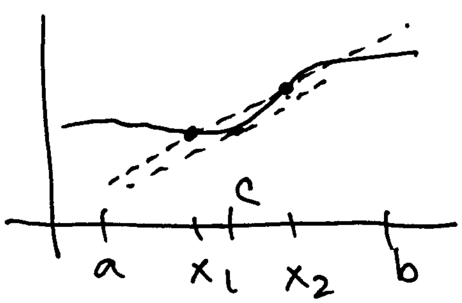


If $f(b) \le f(a)$ then at some point c, $f'(c) \le 0$.

Fact: Positive derivative $\Longrightarrow$ increasing function

Negative derivative $\Longrightarrow$ decreasing function

(3) Only functions $f(x)$ with $f'(x) = 0$ for all x in (a,b) are constant on $[a,b]$.



If f <u>not</u> constant then $f(x_1) \ne f(x_2)$ for some x_1, x_2 and MVT says $f'(c) \ne 0$ some c in (x_1, x_2).

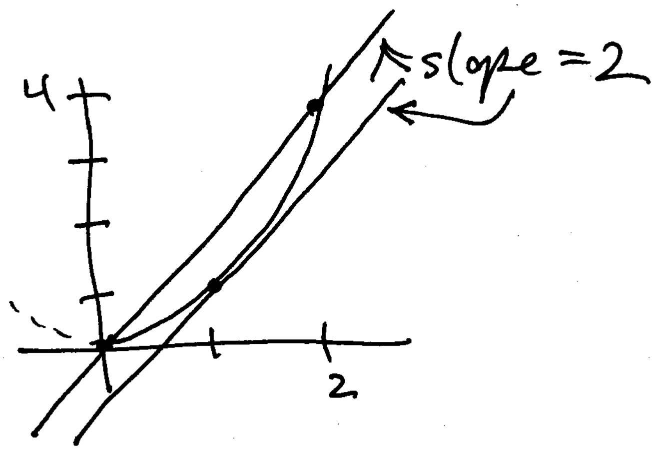
Example of MVT:

$f(x) = x^2 \qquad [0,2]$

Find ~~the~~ a point c satisfying MVT.

Avg rate of change: $\dfrac{f(2)-f(0)}{2-0} = \dfrac{2^2-0^2}{2-0} = \dfrac{4}{2} = 2$

Inst. rate of change: $f'(x) = 2x$



MVT: $2x = 2$

$x = 1$

e.g. $f(x) = \ln(x-1) \qquad [2,4]$

Avg r.o.c.: $\dfrac{f(4)-f(2)}{4-2} = \dfrac{\ln(3) - \cancelto{0}{\ln(1)}}{2} = \frac{1}{2}\ln(3) \approx .5493\ldots$

Inst r.o.c.: $f'(x) = \dfrac{1}{x-1}$

MVT: $\dfrac{1}{x-1} = \frac{1}{2}\ln(3)$

$\dfrac{2}{x-1} = \ln(3)$

$\dfrac{2}{\ln(3)} = x-1$

$x = 1 + \dfrac{2}{\ln(3)} \approx 2.82$

So $2 < 2.82 < 4$

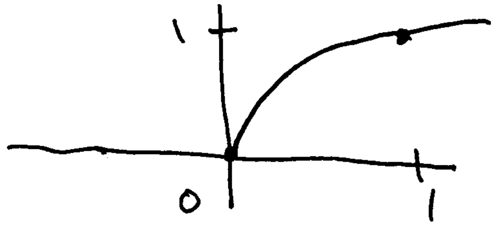
as required.

eg #6) MVT needs:

① $f(x)$ continuous on $[a,b]$

② $f(x)$ differentiable on (a,b)

$f(x) = x^{4/5}$ $\quad [0,1]$. $\qquad$ $f(x)$ cont on $[0,1]$ ✓



$$f'(x) = \frac{4}{5}x^{\frac{4}{5}-1} = \frac{4}{5}x^{-1/5}$$

$$= \frac{4}{5x^{1/5}}$$

undefined at $x=0$

Test: $\dfrac{f(1)-f(0)}{1-0} = \dfrac{1^{4/5}-0^{4/5}}{1-0} = 1$

MVT still OK because $f'(x)$ exists for all x in $(0,1)$.

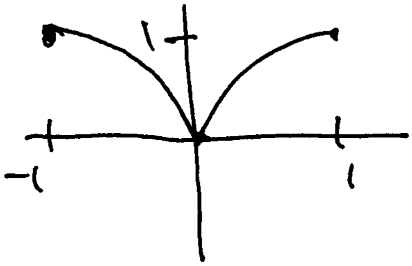
$$f'(x) = \frac{4}{5x^{1/5}}$$

$$\frac{4}{5x^{1/5}} = 1$$

$$x^{1/5} = \frac{4}{5}$$

$$x = \left(\frac{4}{5}\right)^5 = \frac{1024}{3125} \quad \text{(between 0 and 1)}$$

What if we took $f(x) = x^{4/5}$ $\quad [-1,1]$.



$$f(-1) = (-1)^{4/5} = \left[(-1)^4\right]^{1/5}$$

$f(x)$ is cont on $[-1,1]$ ✓ $\qquad = 1^{1/5} = 1$

$f'(x)$ not differentiable on $(-1,1)$ because $f'(0)$ does not exist

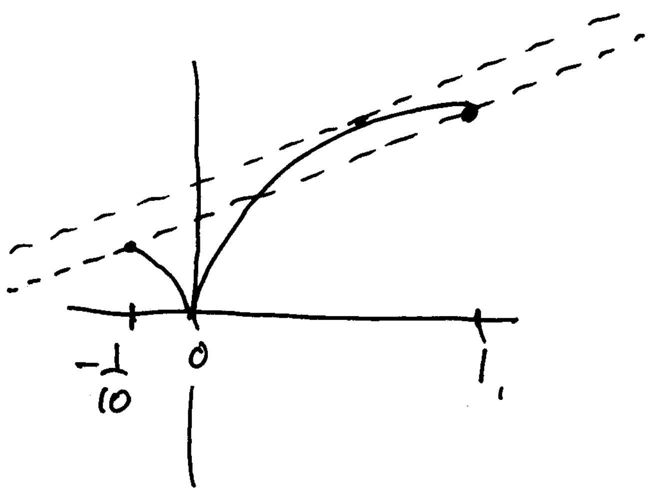
So MVT hypotheses not satisfied

Test MVT:

Avg r.o.c. : $\dfrac{f(1)-f(-1)}{1-(-1)} = \dfrac{1-1}{2} = 0$

Inst r.o.c. : $f'(x) = \dfrac{4}{5x^{1/5}}$ $\quad \dfrac{4}{5x^{1/5}} = 0$ no solution

So MVT fails.



In this case:
Hypotheses of MVT do not hold BUT
Conclusion of MVT holds

eg #16 $\quad f(x) = x^3 + \dfrac{4}{x^2} + 7 \quad (-\infty, 0)$

$f'(x) = 3x^2 - 8x^{-3}$

$f'(x) = 0$:

$3x^2 = 8x^{-3}$

$3x^2 = \dfrac{8}{x^3}$

$x^5 = \dfrac{8}{3}$

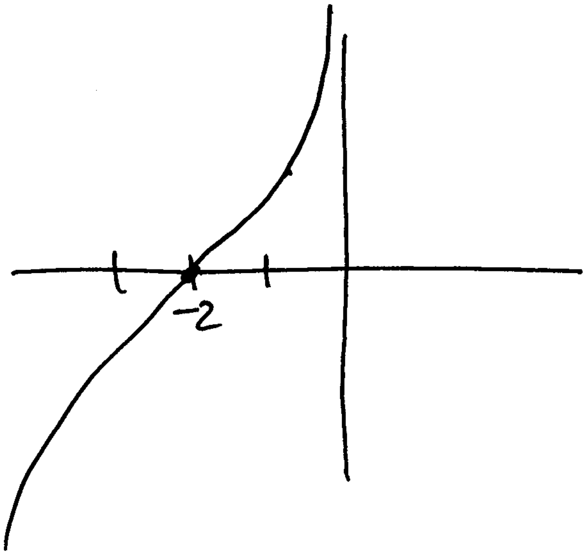
$x = \left(\dfrac{8}{3}\right)^{1/5} \approx 1.2$

f' + + + + + +

−3 −2 −1 0

$f'(-1) = 3 + 8 = 11 > 0$

$\therefore f$ is increasing on $(-\infty, 0)$



$f(-1) = -1 + 4 + 7 = 10$

$f(-2) = -8 + 1 + 7 = 0$

$f(-3) = -27 + \frac{4}{9} + 7 < 0$

So graph crosses x axis <u>exactly</u> once on $(-\infty, 0)$.

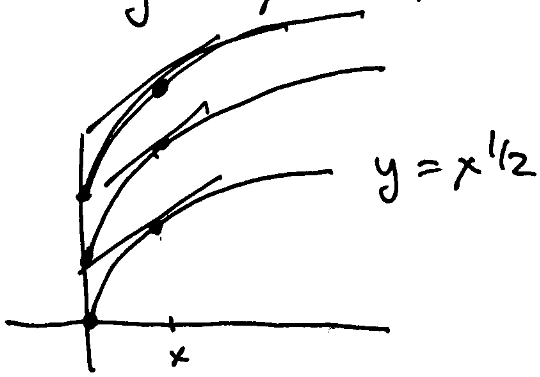
eg #30) a) $y' = \frac{1}{2\sqrt{x}}$ Find all possible y.

$y = \sqrt{x} = x^{1/2}$ $\quad y' = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

$y = x^{1/2} + 3$ $\quad y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Any y with $y' = \frac{1}{2\sqrt{x}}$ has the form

$y = x^{1/2} + C$ (C is any constant).



eg #40) $v(t) = \dfrac{1}{t+2}$ $\quad t > -2 \quad \underline{s(-1) = \frac{1}{2}}$

Find position function $s(t)$.

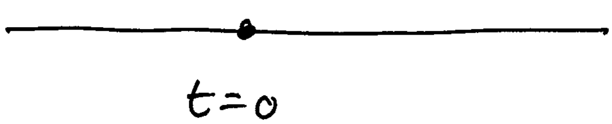
Know $v(t) = s'(t)$

$$s'(t) = \frac{1}{t+2}$$

unknown.

$$s(t) = \underline{\ln(t+2)} + C$$

use $\underline{s(-1) = \frac{1}{2}}$ to find C



$$\frac{1}{2} = s(-1) = \ln(-1+2) + C = \cancelto{0}{\ln(1)} + C = C$$

$\therefore$ $C = \frac{1}{2}$ and $\boxed{s(t) = \ln(t+2) + \frac{1}{2}}$