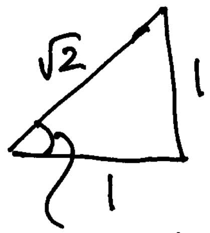
MAPLE #2 due Thursday

Related Rates (cont'd)

$\tan^{-1}(1) = \alpha$



$\alpha = \tan^{-1}(1)$

$\alpha = \frac{\pi}{4}$

$\sin(\tan^{-1}(1)) = 1/\sqrt{2}$

$\cos(\tan^{-1}(1)) = 1/\sqrt{2}$

$\cot(\tan^{-1}(1)) = 1$

$\vdots$

common values for:
$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$

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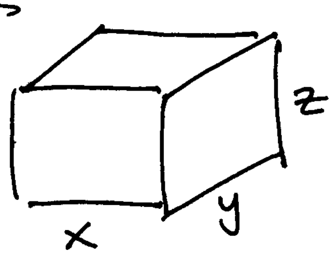


(a) V = volume of box

$V = xyz$

$$\frac{d}{dt}(V) = \frac{d}{dt}(xyz)$$

$$\frac{dV}{dt} = x\frac{d}{dt}(yz) + yz\frac{dx}{dt}$$

$$= x\left(y\frac{dz}{dt} + z\frac{dy}{dt}\right) + yz\frac{dx}{dt}$$

$$= xy\frac{dz}{dt} + xz\frac{dy}{dt} + yz\frac{dx}{dt}$$

Find $\frac{dV}{dt}$ when $\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = -2 \quad \frac{dz}{dt} = 1$

$$x = 4 \quad y = 3 \quad z = 2$$

$$\frac{dV}{dt} = (4)(3)(1) + (4)(2)(-2) + (3)(2)(1)$$

$$= 12 - 16 + 6 = 2 \ m^3/\text{sec.}$$

(c) s = diagonal length

$$s = (x^2+y^2+z^2)^{1/2}$$

$$\frac{d}{dt}(s) = \frac{d}{dt}(x^2+y^2+z^2)^{1/2}$$

$$\frac{ds}{dt} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot \frac{d}{dt}(x^2+y^2+z^2)$$

$$= \frac{\cancel{1}}{\cancel{2}}(x^2+y^2+z^2)^{-1/2} \cdot \left(\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} + \cancel{2}z\frac{dz}{dt}\right)$$

$$\frac{ds}{dt} = (4^2+3^2+2^2)^{-1/2} \cdot (4\cdot 1 + 3(-2) + 2\cdot 1)$$

$$= \frac{1}{(29)^{1/2}} \cdot (4-6+2) = 0 \ \frac{m}{sec}$$

$$s^2 = x^2+y^2+z^2$$

$$\frac{d}{dt}(s^2) = \frac{d}{dt}(x^2+y^2+z^2)$$

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} + \cancel{2}z\frac{dz}{dt}$$

$$s = (4^2+3^2+2^2)^{1/2} = (29)^{1/2}$$

$$(29)^{1/2}\frac{ds}{dt} = (4)(1) + (3)(-2) + (2)(1)$$

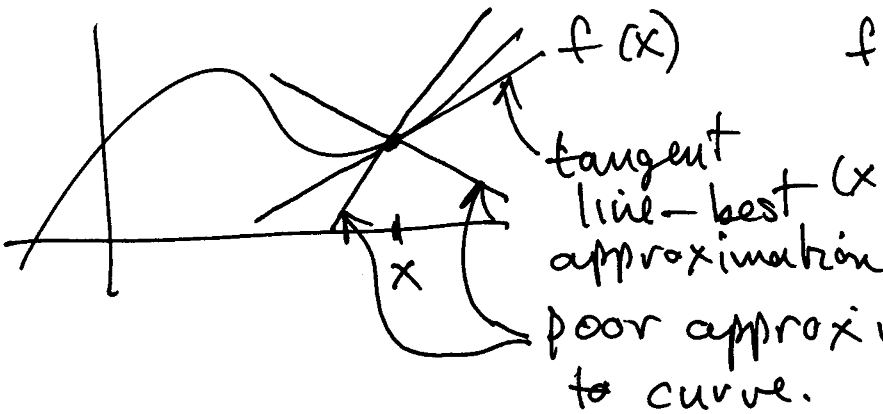
$$29^{1/2}\frac{ds}{dt} = 4 - 6 + 2 = 0$$

$$\frac{ds}{dt} = 0 \ m/\text{sec.}$$

3.10 Linearization + Differentials

Linearization

Idea: Derivative gives: ① slope of tangent line, ② instantaneous rate of change.
Main point is to look at tangent line as the BEST LINEAR APPROXIMATION to $f(x)$ near x.

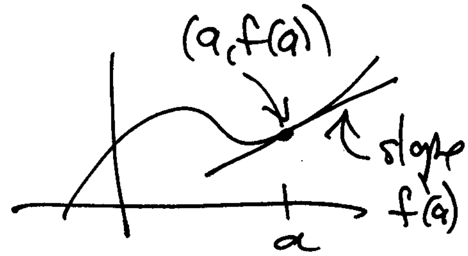


$f'(x)$ – slope of tang. line.

$(x, f(x))$ – point on tangent line.

If you zoom in on curve near $(x, f(x))$ tangent line and curve become indistinguishable.

Given $f(x)$ and point $x = a$
Find equation of tangent line to $f(x)$ at $x = a$.



slope: $f'(a)$ point: $(a, f(a))$

eqn of tangent line:

$$y - f(a) = f'(a)(x - a)$$

$$\boxed{y = f(a) + f'(a)(x - a)}$$

Def: The linearization of $f(x)$ at $x=a$ is

$$L(x) = f(a) + f'(a)(x-a).$$ ($L(x)$ is the equation of the tangent line).

e.g. Find linearization of $f(x) = \sqrt{1+x}$ at $x=0$.

$$f(a) = f(0) = \sqrt{1+0} = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(a) = f'(0) = \frac{1}{2}(1+0)^{-1/2} = \frac{1}{2}$$

$$\boxed{f(x) = (1+x)^{1/2}} \qquad L(x) = 1 + \frac{1}{2}(x-0)$$

$$= \frac{1}{2}x + 1 \ //$$

e.g. $f(x) = (x^2+9)^{1/2}$ at $x=4$

$$f(4) = (4^2+9)^{1/2} = 5$$

$$f'(x) = \frac{1}{\cancel{2}}(x^2+9)^{-1/2}(\cancel{2}x) = x\,(x^2+9)^{-1/2}$$

$$f'(4) = (4)(4^2+9)^{-1/2} = 4\,(25)^{-1/2} = 4 \cdot \frac{1}{5} = \frac{4}{5}$$

$$L(x) = f(4) + f'(4)(x-4)$$

$$= 5 + \frac{4}{5}(x-4)$$

$$= \frac{4}{5}x + \frac{9}{5} \ //$$

Linearizations as approximations

Saw: $f(x) = (1+x)^{1/2}$ then $f(x) \approx L(x)$ if x near 0.

$L(x) = \frac{1}{2}x + 1$

$f(.2) = \sqrt{1.2} = 1.095445115$

$L(.2) = \frac{1}{2}(.2) + 1 = 1.10$

Absolute error: $|f(.2) - L(.2)| \cong .00455$.

Pct error $= \dfrac{\text{Absolute error}}{|\text{true value}|} (\times 100)$

$= \dfrac{|f(.2) - L(.2)|}{|f(.2)|} (\times 100)$

$\cong (.004158)(100) \cong .4\%$

$f(.5) = \sqrt{1.5} \cong 1.2247\ldots$

$L(.5) = \frac{1}{2}(.5) + 1 = 1.25$

Abs error: $|f(.5) - L(.5)| \approx .02525$

Pct error: $\dfrac{|f(.5) - L(.5)|}{|f(.5)|} (100) \approx 2.1\%$

If I want better approx of $f(.5)$ then I could move a closer to $.5$, in fact could take $a = .5$

But $L(x) = \underbrace{f(.5)}_{\text{but this is what I want!}} + \underbrace{f'(.5)}_{\text{hard to compute!}}(x - .5)$

So typically choose a so that $f(a)$ and $f'(a)$ are easy to compute.

e.g. #14 $f(x) = \sin^{-1}(x)$ $\quad x_0 = \frac{\pi}{12} \approx .261799\ldots$

where do we center $L(x)$ (i.e. what is $x = a$?)

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ $\quad$ Try $x = 0$. $\quad f(0) = 0$, $f'(0) = 1$

If choose $x = .2$ (for example) need to find $f'(.2) = \frac{1}{\sqrt{.96}}$ (not easy) and $f(.2) = \sin^{-1}(.2)$?

$L(x) = f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$

$L(\pi/12) = \pi/12 \approx .261799\ldots$

$f(\pi/12) \cong .2648861470$

Pct error: $\frac{|f(\pi/12) - L(\pi/12)|}{|f(\pi/12)|}(100) \approx 1.2\%$ error

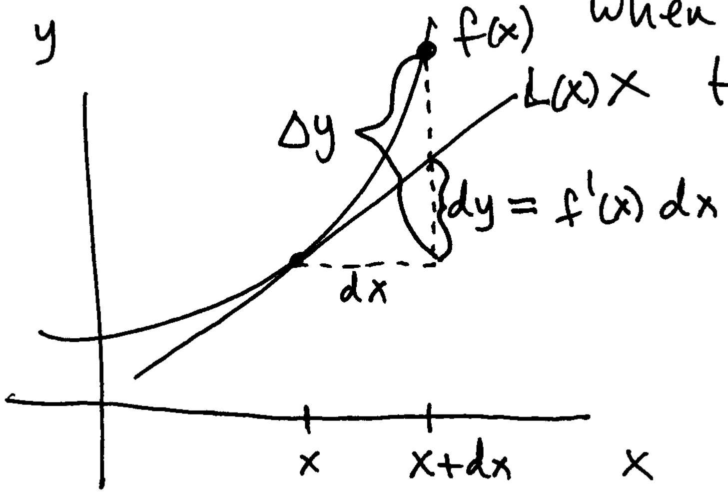
Differential

Exploiting the fact that $L(x)$ (the linearization) is the BEST LINEAR APPROXIMATION to $f(x)$ near $x=a$..

Definition The differential dy of $y=f(x)$ is $dy = f'(x)\,dx$. (Also sometimes say $df = f'(x)\,dx$.)

① dy is a variable dependent on x and dx, i.e. dy - dependent variable, x, dx - independent variables.

② Interpretation: dy = the change in $L(x)$ when x goes from x to $x+dx$.



Since $f(x) \approx L(x)$ then $dy \approx \Delta y$ = the actual change in $f(x)$.

In other words,

$$\Delta y = f(x+dx) - f(x) \quad \text{(actual change in } y)$$

$$dy = f'(x)\,dx \quad \text{(differential or the actual change in } L(x))$$

$$\boxed{dy \approx \Delta y \quad \text{when } dx \text{ is small.}}$$

In still other words.

If $dx = \Delta x$, a change in x,
what will be the change in y, Δy?

$$\Delta y = f(x+\Delta x) - f(x) = f(x+dx) - f(x)$$

So

$$\boxed{\frac{\Delta y}{\Delta x} = \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{difference quotient}} \approx \frac{dy}{dx} = f'(x) \quad \text{if } dx = \Delta x \text{ is small.}}$$

eg #20) p251

$$y = x(1-x^2)^{1/2} \quad \text{Find } dy.$$

$$dy = f'(x)\,dx$$

$$f(x) = x(1-x^2)^{1/2}$$

$$f'(x) = x\frac{d}{dx}(1-x^2)^{1/2} + (1-x^2)^{1/2}\frac{d}{dx}(x)$$

$$= x\left(\cancel{\tfrac{1}{2}}(1-x^2)^{-1/2}(-\cancel{2}x)\right) + (1-x^2)^{1/2}(1)$$

$$= -x^2(1-x^2)^{-1/2} + (1-x^2)^{1/2}$$

$$= \frac{-x^2}{(1-x^2)^{1/2}} + \frac{(1-x^2)^{1/2}}{(1-x^2)^{1/2}} \cdot (1-x^2)^{1/2}$$

$$= \frac{-x^2 + (1-x^2)}{(1-x^2)^{1/2}}$$

$$= \frac{1-2x^2}{(1-x^2)^{1/2}} \;//$$

$$dy = \frac{1-2x^2}{(1-x^2)^{1/2}}\,dx \;//$$

#22) $y = \dfrac{2\sqrt{x}}{3(1+\sqrt{x})}$ Find dy

$$dy = f'(x)\,dx$$

$$f(x) = \frac{2x^{1/2}}{3(1+x^{1/2})}$$

$$f'(x) = \frac{3(1+x^{1/2})(x^{-1/2}) - \cancel{2}x^{1/2}(3(\cancel{\tfrac{1}{2}}x^{-1/2}))}{9(1+x^{1/2})^2}$$

$$= \frac{3x^{-1/2} + \cancel{3} - \cancel{3}}{9(1+x^{1/2})^2} = \frac{\cancel{3}x^{-1/2}}{3\cancel{9}(1+x^{1/2})^2}$$

$$= \frac{1}{3x^{1/2}(1+x^{1/2})^2} \qquad dy = \frac{1}{3x^{1/2}(1+x^{1/2})^2}\,dx$$

eg #40) $f(x) = 2x^2 + 4x - 3 \quad x_0 = -1 \quad dx = 0.1$

Find $\Delta f = f(x_0 + dx) - f(x_0)$

$\underline{x_0 = -1} \quad \underline{dx = .1} \qquad x_0 + dx = -.9$

$$\Delta f = f(-.9) - f(-1)$$

$$= (2(-.9)^2 + 4(-.9) - 3) - (2 - 4 - 3)$$

$$= 2(.81) - 3.6 - 3 + 5$$

$$= 1.62 - 3.6 + 2 \quad = .02$$

Find $df = f'(x_0)\,dx$, $f'(x) = 4x + 4$

$$df = (4(-1) + 4)(.1) = 0$$

Error : $|\Delta f - df| = .02$ (fairly small)

#44) $f(x) = x^3 - 2x + 3 \qquad x_0 = 2 \quad dx = .1$

$\Delta f = f(x_0 + dx) - f(x_0) \qquad x_0 + dx = 2.1$

$= f(2.1) - f(2)$

$= \left((2.1)^3 - 2(2.1) + 3\right) - (8 - 4 + 3)$

$= 9.261 - 4.2 + 3 - 7$

$= 9.261 - 8.2 = 1.061$

$df = f'(x_0)\,dx = f'(2)\,(.1)$

$f'(x) = 3x^2 - 2$

$f'(2) = 3 \cdot 4 - 2 = 10$

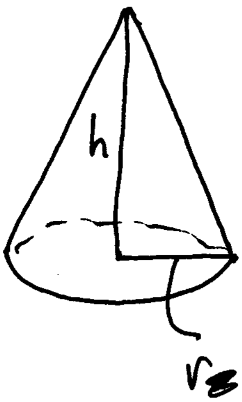
$df = f'(2)(.1) = 10\,(.1) = 1.0$

Can see $df \approx \Delta f$

Error: $|\Delta f - df| = .061$

Pct error: $\dfrac{.061}{1.061} \cong 5.7\%$

#48)



$$S = \pi r (r^2+h^2)^{1/2}$$

If r changes from r_0 to $r_0 + dr$, how does S change?

Exact answer: $$\Delta S = \pi (r_0+dr)\left((r_0+dr)^2+h^2\right)^{1/2} - \pi (r_0)(r_0^2+h^2)^{1/2}$$

Approx. via differentials, i.e. the "differential formula" asked for in problem.

$$dS = S'(r_0)\,dr$$

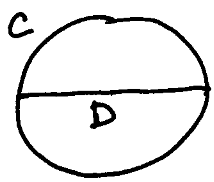
$$S'(r) = \pi r \frac{d}{dr}(r^2+h^2)^{1/2} + (r^2+h^2)^{1/2}\frac{d}{dr}(\pi r)$$

$$= \pi r\left(\frac{1}{\cancel{2}}(r^2+h^2)^{-1/2}(\cancel{2}r)\right) + (r^2+h^2)^{1/2}\cdot\pi$$

$$= \pi r^2 (r^2+h^2)^{-1/2} + \pi (r^2+h^2)^{1/2}$$

$$\therefore\ dS = \left(\pi r_0^2 (r_0^2+h^2)^{-1/2} + \pi (r_0^2+h^2)^{1/2}\right) dr$$

#52)



D = diameter in inches

C = circumference in inches

$$C = \pi D$$

$$dC = C'(D) \cdot dD = \pi \, dD$$

In example $D = 10$, $dC = 2$, $dD = ?$

$$2 = \pi \, dD \qquad \boxed{dD = \frac{2}{\pi} \text{ inches}}$$

Cross-sectional area: $A = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D^2$

$$dA = A'(D)\, dD = \left(\frac{\pi}{2} D\right) dD$$

In example: $D = 10$, $dD = \frac{2}{\pi}$

$$\therefore \; dA = \left(\frac{\pi}{2}\right)(10)\left(\frac{2}{\pi}\right) = 10 \text{ in}^2.$$

$$\boxed{\begin{aligned} \Delta A &= \frac{\pi}{4}\left(10 + \frac{2}{\pi}\right)^2 - \frac{\pi}{4}(10)^2 \\ &\approx 10.32 \text{ in}^2 \end{aligned}}$$