Inverse Trig Functions

$$f^{-1}(x) = y \iff f(y) = x$$

$$\ln(x) = y \iff e^y = x$$

$$\log_a(x) = y \iff a^y = x$$

$y = x^r$ r is any number

Know that if $r = 0, \pm 1, \pm 2, \ldots$

$$\frac{dy}{dx} = r x^{r-1}$$

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

Idea $y = x^r$ then $y = e^{\ln(x^r)} = e^{r \ln(x)}$

$$\frac{dy}{dx} = \underbrace{e^{r\ln(x)}}_{x^r} \cdot \frac{d}{dx}(r \ln(x)) = x^r \cdot r \cdot \frac{1}{x} = r x^{r-1}$$

$$\frac{d}{dx}(x^r) = r x^{r-1} \quad \text{for any } r.$$

$$\boxed{\frac{d}{dx} u^r = r u^{r-1} \frac{du}{dx}, \quad \text{any } r}$$

eg $y = 2x^2 + 3x^{-1/2} + x^{-5/2}$

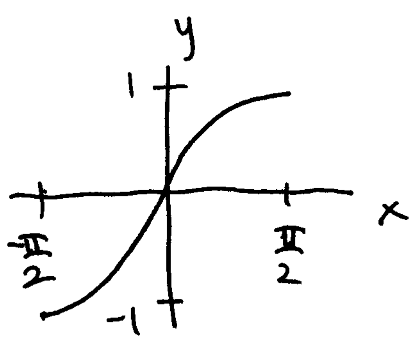
$$y' = 4x + 3\left(-\tfrac{1}{2}x^{-3/2}\right) + \left(-\tfrac{5}{2}x^{-7/2}\right)$$

$$= 4x - \tfrac{3}{2}x^{-3/2} - \tfrac{5}{2}x^{-7/2}.$$

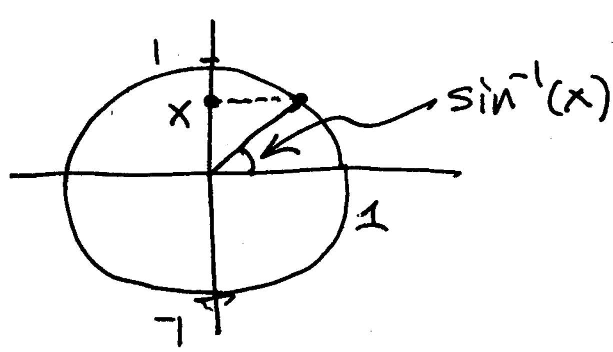
Back to inverse trig functions.

$\sin^{-1}(x)$: $\quad \sin^{-1}(x) = y \iff \sin(y) = x$

$-1 \le x \le 1 \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

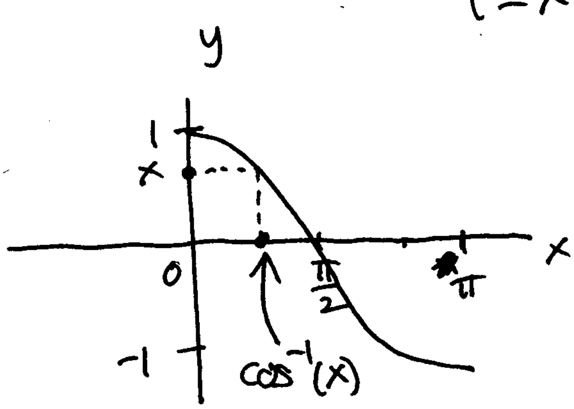


sin(x)

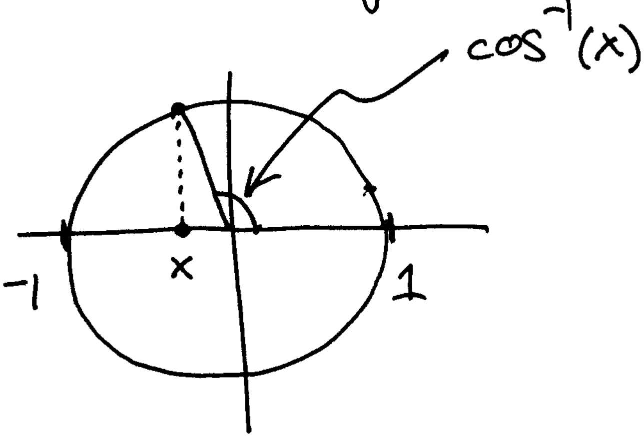


$\cos^{-1}(x)$: $\quad \cos^{-1}(x) = y \iff \cos(y) = x$

$-1 \le x \le 1 \qquad 0 \le y \le \pi$

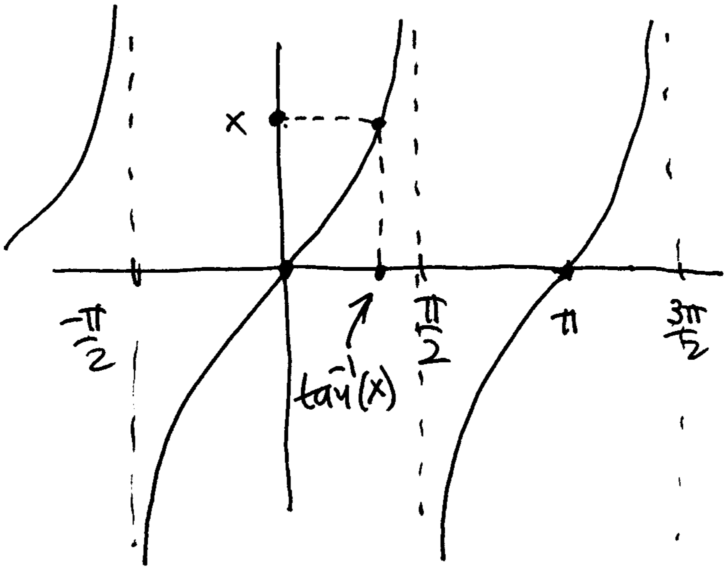


cos(x)



$\tan^{-1}(x)$: $\quad \tan^{-1}(x) = y \iff \tan(y) = x$

$-\infty < x < \infty \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$



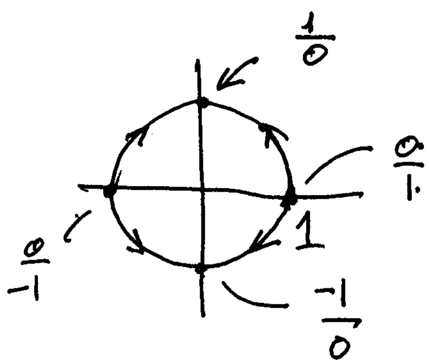
e.g

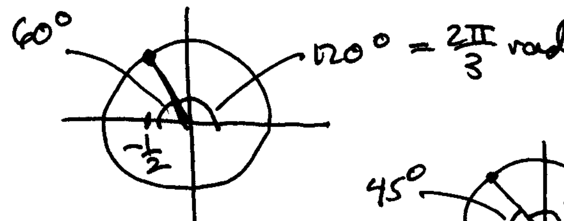
#6) $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

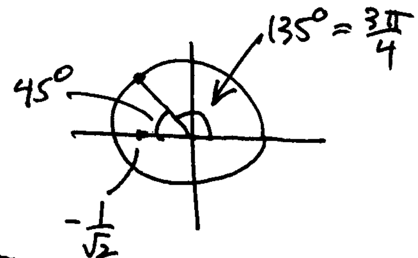
$\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

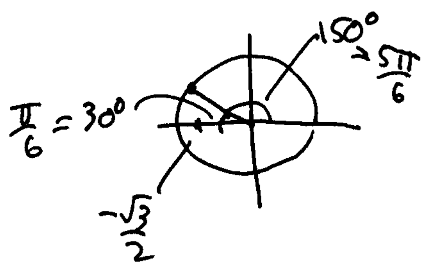
$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$









eg #14) $\underbrace{\alpha = \tan^{-1}\left(\frac{4}{3}\right)}$ Find $\sin(\alpha)$, $\cos(\alpha)$, etc...

$\tan(\alpha) = \frac{4}{3}$

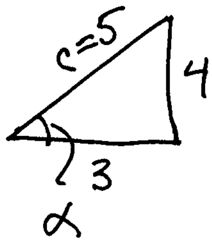
$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\sin(\alpha) = \frac{4}{5}$ $\cos(\alpha) = \frac{3}{5}$

$\cot(\alpha) = \frac{3}{4}$ $\sec(\alpha) = \frac{5}{3}$

$\csc(\alpha) = \frac{5}{4}$

reference triangle

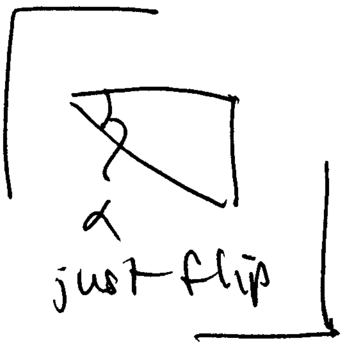


$3^2 + 4^2 = c^2$

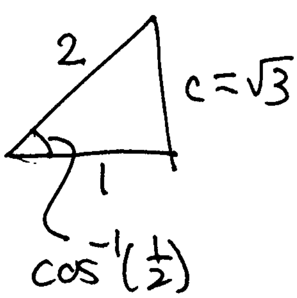
$9 + 16 = c^2$

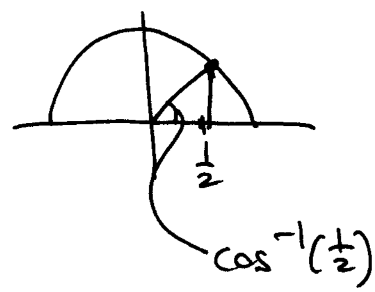
$25 = c^2$

$5 = c$



eg #18) $\sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$





$1^2 + c^2 = 2^2$

$1 + c^2 = 4$

$c^2 = 3$

$c = \sqrt{3}$

$\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

$\tan\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \sqrt{3}$

$\sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = 2$

Derivatives

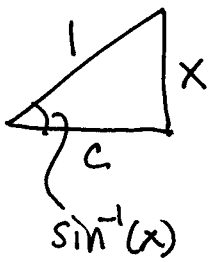
$\sin^{-1}(x)$:

$$\sin(\sin^{-1}(x)) = x$$

$$\frac{d}{dx}\sin(\sin^{-1}(x)) = \frac{d}{dx}(x)$$

$$\cos(\sin^{-1}(x)) \cdot \frac{d}{dx}\sin^{-1}(x) = 1$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}$$

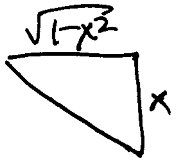


$$c^2 + x^2 = 1^2$$

$$c^2 = 1 - x^2$$

$$c = \sqrt{1-x^2}$$

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$$



still ok

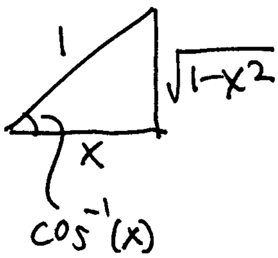
$\cos^{-1}(x)$:

$$\cos(\cos^{-1}(x)) = x$$

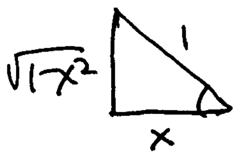
$$\frac{d}{dx}\cos(\cos^{-1}(x)) = \frac{d}{dx}(x)$$

$$-\sin(\cos^{-1}(x))\,\frac{d}{dx}\cos^{-1}(x) = 1$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sin(\cos^{-1}(x))} = \frac{-1}{\sqrt{1-x^2}}$$



$$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$$



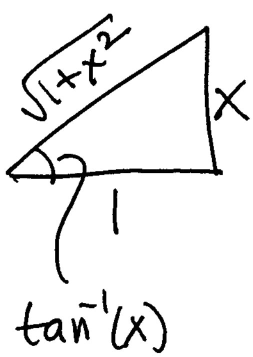
still <u>OK</u>

$\tan^{-1}(x)$:

$$\tan(\tan^{-1}(x)) = x$$

$$\sec^2(\tan^{-1}(x)) \cdot \frac{d}{dx}\tan^{-1}(x) = 1$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{1+x^2}$$



$$\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$$

$$\sec^2(\tan^{-1}(x)) = 1+x^2$$

eg

$$y = \cos^{-1}\left(\frac{1}{x}\right)$$

$$y' = \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2} = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$\boxed{\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = \frac{-1}{x^2}}$$

$$= \cancel{\frac{1}{x^2\sqrt{1-\left(\frac{1}{x^2}\right)}}}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{x^2 \frac{\sqrt{x^2-1}}{x}} = \frac{1}{x\sqrt{x^2-1}} \quad //$$

$$y = \tan^{-1}(\ln(t))$$

$$y' = \frac{1}{1+[\ln(t)]^2} \frac{d}{dt}\ln(t) = \frac{1}{1+\ln^2(t)} \cdot \frac{1}{t} = \frac{1}{t\left(1+\ln^2(t)\right)} \quad //$$

3.9 Related Rates

Idea: If two quantities are related then their rates of change are related.

#2, p236 S = surface area of ball

r = radius of ball

$$S = 4\pi r^2$$

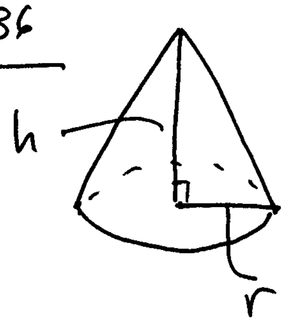
S, r are functions of t (think of "time")

$\therefore$ $\frac{dS}{dt}$ and $\frac{dr}{dt}$ are related. How?

$$\frac{d}{dt}(S) = \frac{d}{dt}(4\pi r^2)$$

$$\frac{dS}{dt} = 4\pi\left(2r\frac{dr}{dt}\right) = 8\pi r\frac{dr}{dt}$$

#4, p 236



V = volume of cone

r = radius of base

h = height of cone

$$V = \frac{1}{3}\pi r^2 h$$

Assume V, r, h change with t (time).

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + h\cdot\frac{d}{dt}(r^2)\right)$$

$$= \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{1}{3}\pi h\left(2r\frac{dr}{dt}\right)$$

$$= \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{2}{3}\pi r h\frac{dr}{dt}$$

a. r constant means $\frac{dr}{dt} = 0$

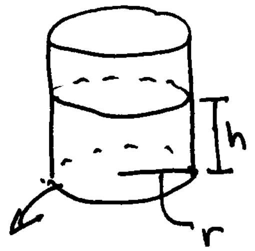
$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt}$$

b. h constant means $\frac{dh}{dt} = 0$

$$\frac{dV}{dt} = \frac{2}{3}\pi r h\frac{dr}{dt}$$

c. $\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{2}{3}\pi r h\frac{dr}{dt}$.

eg 1 p233



h = height of water in m.

r = radius of tank in m.

V = volume of water in L

Relationship between V, r, h:

$$1000\ L = 1\ m^3$$

$$V = \underbrace{1000\, \underbrace{\pi r^2 h}_{\text{vol in } m^3}}_{\text{vol in L.}}$$

Relationship between $\frac{dV}{dt}$, $\frac{dh}{dt}$ (note: r is constant)

$$\frac{d}{dt}(V) = \frac{d}{dt}(1000\pi r^2 h)$$

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}$$

Given $\frac{dV}{dt} = -3000$ L/min

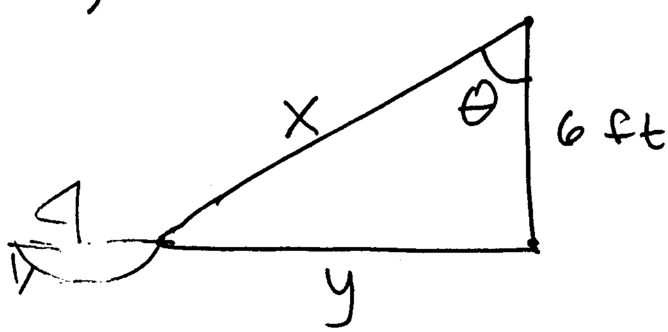
r = not given explicitly

$\frac{dh}{dt}$ is what we want

$$-3000 = 1000\pi r^2 \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{-3}{\pi r^2} \text{ m/min.}$$

#22)



x = length of rope in feet

y = dist from dock in feet

θ = angle as shown.

a. $\frac{dx}{dt} = -2$ ft/~~min~~sec Want: $\frac{dy}{dt}$

Relationship between x and y.

$$y^2 + 36 = x^2$$

$$\frac{d}{dt}(y^2+36) = \frac{d}{dt}(x^2)$$

$$\cancel{2}y\frac{dy}{dt} = \cancel{2}x\frac{dx}{dt}$$

$$8 \cdot \frac{dy}{dt} = 10 \cdot (-2)$$

$$\frac{dy}{dt} = \frac{-20}{8} = -2.5 \text{ ft/sec.}$$

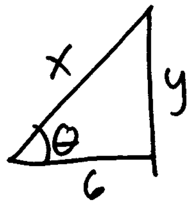
$x = 10$ ft

$y = 8$ ft

$$y^2 + 36 = 10^2$$
$$y^2 + 36 = 100$$
$$y^2 = 64$$
$$y = 8$$

b. Want: $\frac{d\theta}{dt}$

Relationship between x and θ



$$\sin\theta = \frac{y}{x}$$

$$\frac{d}{dt}(x\sin\theta = y)$$

$$x\cos\theta\left(\frac{d\theta}{dt}\right) + \sin\theta\cdot\frac{dx}{dt} = \frac{dy}{dt}$$

$$x = 10 \quad \frac{dx}{dt} = -2 \quad \frac{dy}{dt} = -2.5 \quad \sin\theta = \frac{4}{5} \quad \cos\theta = \frac{3}{5}$$

$$\cancel{10}^{2}\cdot\frac{3}{\cancel{5}}\cdot\frac{d\theta}{dt} + \frac{4}{5}\cdot(-2) = -2.5$$

$$6\,\frac{d\theta}{dt} = -\frac{5}{2} + \frac{8}{5} = -\frac{9}{10}$$

$$\frac{d\theta}{dt} = -\frac{\cancel{9}^{3}}{10}\cdot\frac{1}{\cancel{6}_{2}} = -\frac{3}{20}\ \text{rad/sec.}$$

$$\cos\theta = \frac{6}{x} \qquad x\cos\theta = 6$$

$$\frac{d}{dt}(x\cos\theta) = \frac{d}{dt}(6)$$

$$x\left(-\sin\theta\,\frac{d\theta}{dt}\right) + \cos\theta\cdot\frac{dx}{dt} = 0$$

$$-x\sin\theta\left(\frac{d\theta}{dt}\right) + \cos\theta\,\frac{dx}{dt} = 0$$

$x = 10 \quad \frac{dx}{dt} = -2 \quad \sin\theta = \frac{4}{5} \quad \cos\theta = \frac{6}{10} = \frac{3}{5}$

$$-\cancel{10}^{2} \cdot \frac{4}{\cancel{5}} \frac{d\theta}{dt} + \frac{3}{5}(-2) = 0$$

$$-8 \frac{d\theta}{dt} = \frac{6}{5}$$

$$\frac{d\theta}{dt} = \frac{-\cancel{6}^{3}}{5} \cdot \frac{1}{\cancel{8}_{4}} = \frac{-3}{20} \text{ rad/sec.}$$