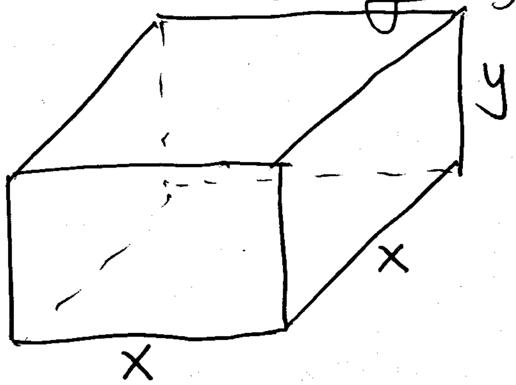
Quiz 10 - Wednesday - Sections 3.5, 4.1

3.5 18)



Want to find x and y that minimize C.

C = cost of the box

C = (cost of top + bottom) + (cost of sides)

$= \underset{\text{bottom}}{2x^2} + \underset{\text{top}}{2x^2} + \underset{\text{\$1 per sq. meter}}{(1)}\underset{\text{area of sides}}{(4xy)}$

$= 4x^2 + 4xy$

Constraint: $250 = x^2 y \rightarrow y = \frac{250}{x^2}$

$C = 4x^2 + 4x\left(\frac{250}{x^2}\right)$

$= 4x^2 + \frac{1000}{x}$ ← MINIMIZE THIS

$$\boxed{\begin{aligned} C &= 4x^2 + 1000x^{-1} \\ C' &= 8x - 1000x^{-2} \\ &= 8x - \frac{1000}{x^2} \end{aligned}}$$

Find critical numbers:

$C' = 8x - \frac{1000}{x^2}$

$8x - \frac{1000}{x^2} = 0$

$8x = \frac{1000}{x^2}$ → $x^3 = \frac{1000}{8} = 125$

$\underline{x = 5}$

Are we done? <u>NO</u>

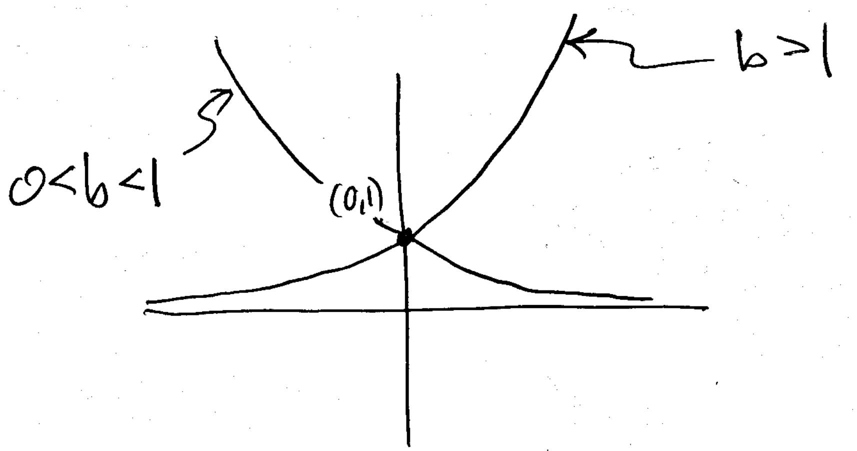
We need to find C.

$C(5) = 4(5)^2 + \frac{1000}{5} = 100 + 200 = 300$

No. We need at least \$300 to make the box.

Exponents.

$f(x) = b^x \quad b > 0,\ b \neq 1$



The natural exponential base

The natural exponential base is the number e defined by

?

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
$$\approx 2.71828\ldots$$

Compound interest

P = principal r = interest rate (annual)

$P_0 = P$ P_k = balance after k years.

$P_1 = P(1+r)$

$P_2 = P(1+r)(1+r) = P(1+r)^2$

$P_3 = P(1+r)^3$

$\vdots$

$P_m = P(1+r)^m$

Say you compound quarterly.

$$P_0 = P$$
$$P_{1/4} = P\left(1+\frac{r}{4}\right)$$
$$P_{2/4} = P\left(1+\frac{r}{4}\right)^2$$
$$P_{3/4} = P\left(1+\frac{r}{4}\right)^3$$
$$P_1 = P\left(1+\frac{r}{4}\right)^4$$

$P = 1000$

$r = .06$

Compound yearly

$$P_1 = 1000(1.06)$$
$$= \$1060$$

$$P_1 = 1000\left(1+\frac{.06}{4}\right)^4$$
$$= 1000(1.06136\cdots)$$
$$= \$1061.36$$

Monthly:

$$P_0 = P$$
$$P_{1/12} = P\left(1+\frac{r}{12}\right)$$
$$P_{2/12} = P\left(1+\frac{r}{12}\right)^2$$
$$\vdots$$
$$P_1 = P\left(1+\frac{r}{12}\right)^{12}$$

$$P_1 = 1000\left(1+\frac{.06}{12}\right)^{12}$$
$$= 1000(1.0616778\ldots)$$
$$\doteq \$1061.68$$

If we compound <u>k times per year</u>

then after t years the balance is:

$$\boxed{B(t) = P\left(1+\frac{r}{k}\right)^{kt}}$$

We see that if k is large, then $B(t)$ gets larger as well.

What if we let $k \to \infty$?

$$B(t) = P\left[\left(1+\frac{r}{k}\right)^{\frac{k}{r}}\right]^{rt}$$

say $r = .06$

2.7101715...

2.7142155...

2.717466...

2.7181187...

$\Downarrow$

e

$$\boxed{\text{Continuous compounding: } B(t) = Pe^{rt}}$$

4.2. Logarithmic Functions

If x is a positive number, then the logarithm of x to the base $b (b > 0, b \neq 1)$, denoted $\log_b x$, is the number y such that $b^y = x$; that is,

$$y = \log_b x \quad \text{if and only if} \quad b^y = x \quad \text{for } x > 0$$

Example
Evaluate $\log_{10} 1,000$.

$\log_{10} 1000 = y$

$10^y = 1000$

$y = 3$

Example
Solve the equation $\log_4 x = \frac{1}{2}$.

$4^{1/2} = x$

$x = 2$

$2^x = 16$

$2^x = 2^4$

$x = 4$

$x = \log_2(16)$

Properties of Logarithms

$b^0 = 1$

Let $b(b > 0, b \neq 1)$ be any logarithmic base. Then,

$$\log_b 1 = 0 \quad \text{and} \quad \log_b b = 1 \Longleftrightarrow b^1 = b$$

and if u and v are any positive numbers, then

- The equality rule: $\log_b u = \log_b v$ if and only if $u = v$
- The product rule: $\log_b(uv) = \log_b u + \log_b v$
- The power rule: $\boxed{\log_b u^r = r \log_b u}$ for any real number r
- The quotient rule: $\log_b \left(\frac{u}{v}\right) = \log_b u - \log_b v$
- The inversion rule: $\log_b b^u = u$

$$\log_b(b^u) = u \log_b(b) = u$$

Properties of Logarithms

Example

Use logarithm rules to rewrite each of the following expressions in terms of $\log_3 2$ and $\log_3 5$.

a. $\log_3 270$ $= \log_3(27 \cdot 10) = \log_3(3 \cdot 3 \cdot 3 \cdot 2 \cdot 5)$

$= \log_3(3) + \log_3(3) + \log_3(3) + \log_3(2) + \log_3(5)$

$= 3 + \log_3(2) + \log_3(5)$

b. $\log_3\left(\frac{64}{125}\right)$

$= \log_3(64) - \log_3(125)$

$= \log_3(2^6) - \log_3(5^3)$

$= 6\log_3(2) - 3\log_3(5)$

Properties of Logarithms

Example

Use logarithm rules to simplify each of the following expression.

a. $\log_3(x^3y^{-4})$ $= \log_3(x^3) + \log_3(y^{-4})$

$= 3\log_3(x) - 4\log_3(y)$

b. $\log_7(x^3\sqrt{1-y^2})$

$= \log_7(x^3) + \log_7((1-y^2)^{1/2})$

$= 3\log_7(x) + \frac{1}{2}\log_7(1-y^2)$

$= 3\log_7(x) + \frac{1}{2}\log_7((1-y)(1+y))$

$= 3\log_7(x) + \frac{1}{2}\log_7(1-y) + \frac{1}{2}\log_7(1+y)$ //

$\log_3(x^3y^{-4}) = 5$

$3\log_3(x) - 4\log_3(y) = 5$

$3\log_3(x) - 5 = 4\log_3(y)$

$\frac{1}{4}(3\log_3(x) - 5) = \log_3(y)$

$y = 3^{(\frac{1}{4}(3\log_3(x) - 5))}$

$(\log_3(y) = t \Leftrightarrow 3^t = y)$

$3^5 = x^3y^{-4}$

$243 = x^3y^{-4}$

$y^{-4} = \frac{243}{x^3}$

$y^4 = \frac{x^3}{243}$

$y = \boxed{\left(\frac{x^3}{243}\right)^{1/4}}$

The Natural Logarithm

The logarithm $\log_e x$ is called the natural logarithm of x and is denoted by $\ln x$; that is,

$$y = \ln x \quad \text{if and only if} \quad e^{y} = x$$

Properties of the Natural Logarithm

For positive numbers u and v,

- The equality rule: $\ln u = \ln v$ if and only if $u = v$
- The product rule: $\ln(uv) = \ln u + \ln v$
- The power rule: $\ln u^r = r \ln u$ for any real number r
- The quotient rule: $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$
- Special values: $\ln 1 = 0$ and $\ln e = 1$

The Natural Logarithm

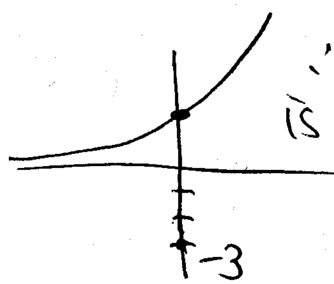
The Inverse Relationship between e^x and $\ln x$

$e^{\ln x} = x$ for $x > 0$ and $\ln e^x = x$ for all x

$\ln(-3) = y$

$\Updownarrow$

$e^y = -3$ NO SOLUTION



$\therefore \ln(x)$ is undefined if $x \leq 0$

Domain of $\ln(x)$ is $x > 0$.

Example

Solve the following equations.

a. $-2 \ln x = 3$

b. $\ln x = 2(\ln 3 - \ln 5)$

c. $\dfrac{5}{1 + 2e^{-x}} = 3$

$5 = 3(1 + 2e^{-x})$

$2 = 6e^{-x}$

$e^{-x} = \frac{1}{3}$

$\ln(e^{-x}) = \ln(\frac{1}{3})$

$-x = \ln(\frac{1}{3})$

$x = -\ln(\frac{1}{3})$

$\ln(x) = -\frac{3}{2}$

$e^{\ln(x)} = e^{-3/2}$

$x = e^{-3/2}$

$\ln(x) = 2(\ln 3 - \ln 5)$

$= 2 \ln(\frac{3}{5})$

$= \ln((\frac{3}{5})^2)$

$e^{\ln(x)} = e^{\ln((3/5)^2)}$

$x = (\frac{3}{5})^2 = \frac{9}{25}$.