Relative Extrema

Example

Find all critical numbers of the function

$$f(x) = x\sqrt{4-x} = x(4-x)^{1/2}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

$$f'(x) = x \cdot \tfrac{1}{2}(4-x)^{-1/2}(-1) + (4-x)^{1/2}$$

$$= -\tfrac{1}{2}x(4-x)^{-1/2} + (4-x)^{1/2}$$

$$= \frac{-x}{2(4-x)^{1/2}} + (4-x)^{1/2}$$

$$= \frac{-x}{2(4-x)^{1/2}} + \frac{2(4-x)^{1/2}(4-x)^{1/2}}{2(4-x)^{1/2}}$$

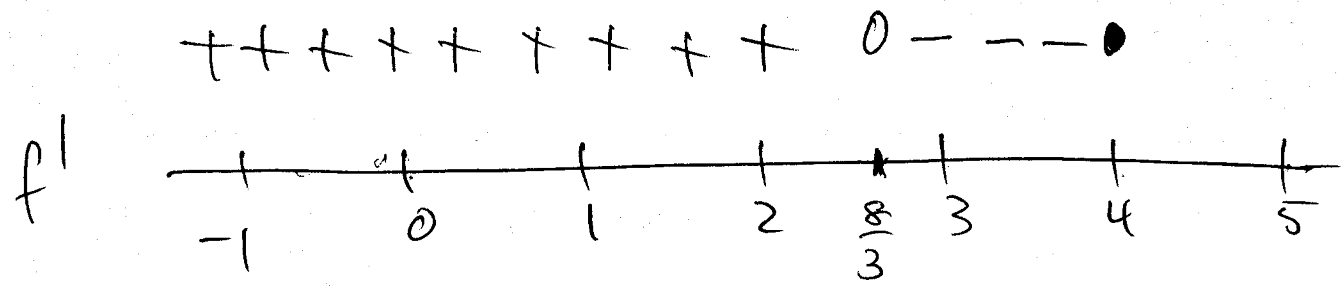
$$= \frac{-x+2(4-x)}{2(4-x)^{1/2}} = \frac{-3x+8}{2(4-x)^{1/2}} \quad //$$

Critical numbers: $x = \frac{8}{3}$ $\quad x = 4$

Increasing / Decreasing
$f' > 0$ $\quad$ $f' < 0$

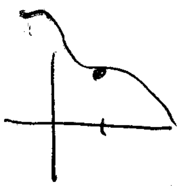
Relative Maxima / Minima
occur where $f' = 0$ or f' undefined

Example (continued)



$$f'(0) = \frac{8}{4} = 2 > 0 \qquad f'(3) = \frac{-1}{2} < 0$$

$$(-1)^{1/2} = \sqrt{-1} \text{ not defined}$$



Critical points: $f\left(\frac{8}{3}\right) = \frac{8}{3}\left(4 - \frac{8}{3}\right)^{1/2}$

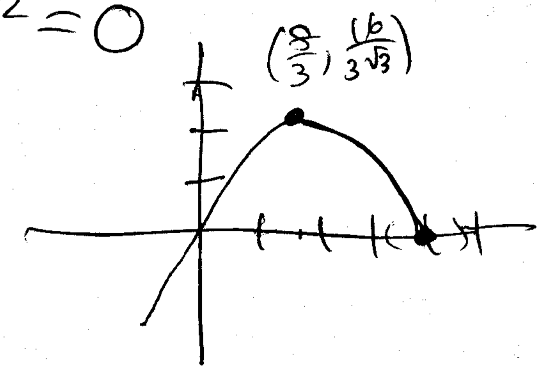
$$= \frac{8}{3}\left(\frac{4}{3}\right)^{1/2} = \frac{16}{3\sqrt{3}}$$

$$f(4) = 4(4-4)^{1/2} = 4(0)^{1/2} = 0$$

$\left(\frac{8}{3}, \frac{16}{3\sqrt{3}}\right)$ $\quad$ $(4, 0)$

$\uparrow$ $\quad$ $\uparrow$

rel. max. $\quad$ relative min.



Sketching the graph

Prodecure for sketching the graph of a continuous function using the derivative

Step 1. Determine the domain of $f(x)$. Set up a number line restricted to include only those numbers in the domain.

Step 2. Find $f'(x)$ and mark each critical number on the restricted number line. Then analyze the sign of $f'(x)$ to determine the intervals of increase and decrease for $f(x)$.

Step 3. For each critical number c, find $f(c)$ and plot the critical point $P(c, f(c))$ on a plane. Plot intercepts and other key points that can be easily found.

Step 4. Sketch the graph of f as a smooth curve joining the critical points in such a way that it rises where $f'(x) > 0$, falls where $f'(x) < 0$, and has a horizontal tangent where $f'(x) = 0$.

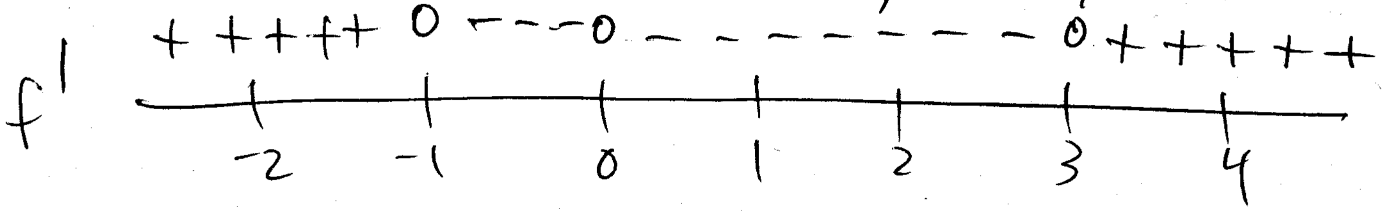
Sketching the graph

Example

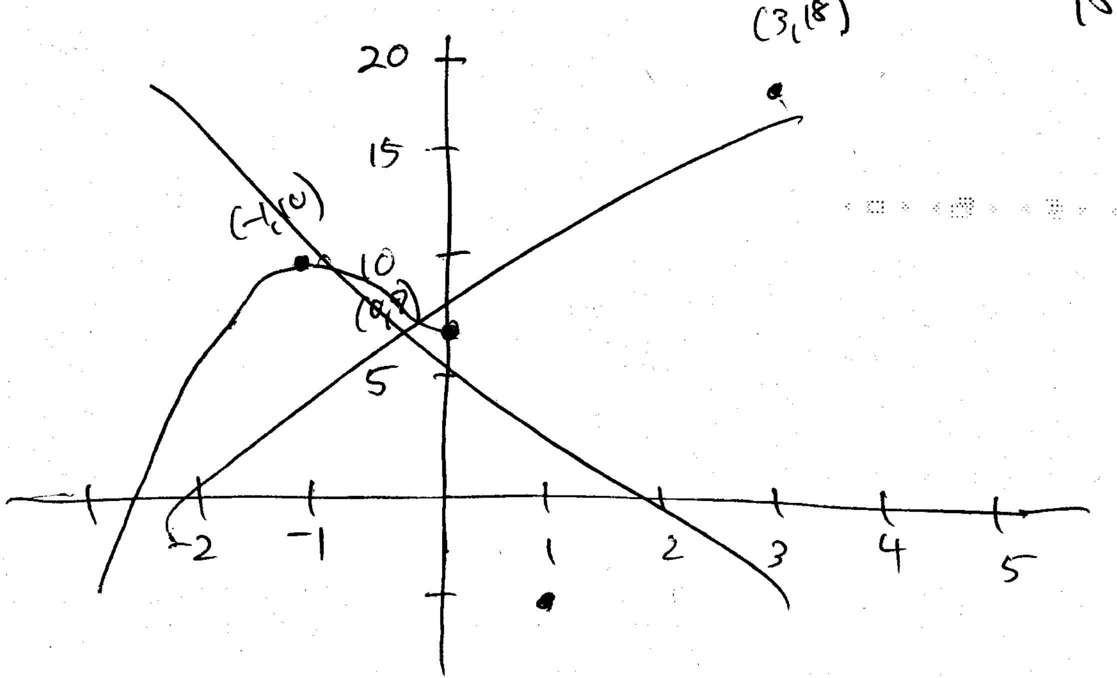
Use calculus to sketch the graph of

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

Critical numbers $x = 0$, $x = -1$, $x = 3$



Critical points: (0, 7) (-1, 10) (3, ~~18~~)
-182



Relative Extrema

Example

Find all critical numbers of the function

$3^5 = 81 \cdot 3 = 243$
486

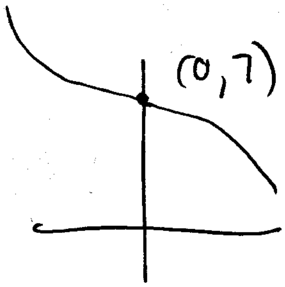
$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Critical numbers: $x=0 \quad x=3 \quad x=-1$

Critical points: $(0,7) \quad (3,\overset{-182}{\cancel{18}}) \quad (-1,10)$

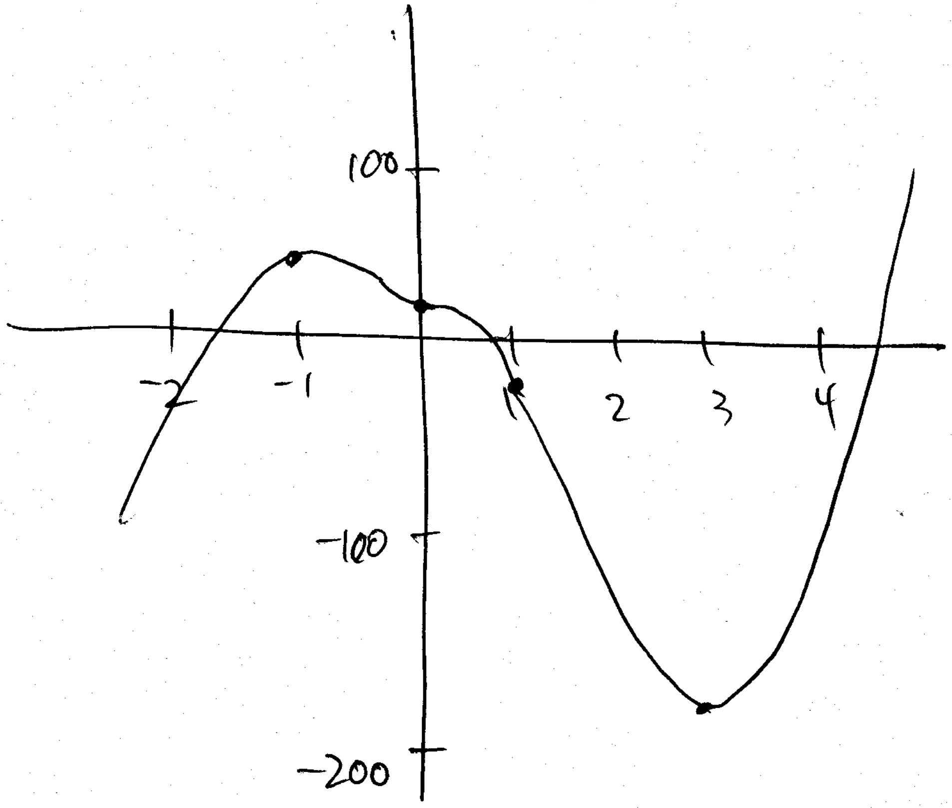
$f(3) = \overset{486}{\cancel{686}} - 405 - 270 + 7 = \cancel{18} = \boxed{-182}$



(0,7) neither

(3,18) relative minimum

(-1,10) relative maximum


100
-100
-200
-2
-1
1
2
3
4

Sketching the graph

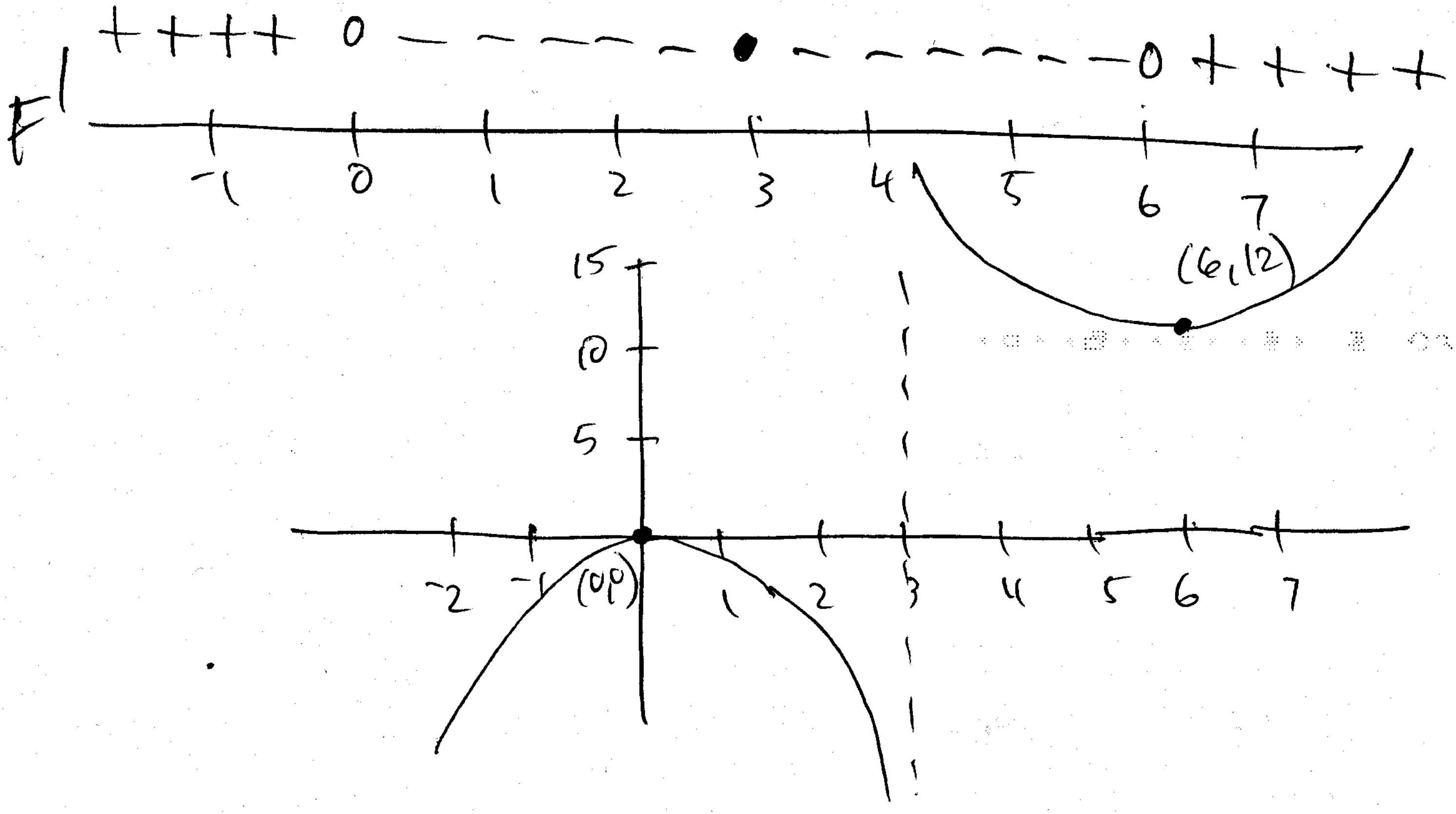
Example

Use calculus to sketch the graph of

$$F(x) = \frac{x^2}{x-3}.$$

Critical numbers: $x=0$ $x=3$ $x=6$

Critical points: $(0,0)$ $(6,12)$



Sketching the graph

Example

Use calculus to sketch the graph of

$$f(x) = \frac{x+1}{x^2+x+1}.$$

Domain: $x^2+x+1=0$ $\qquad x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

never zero.

$$x^2+3x+1$$

Critical points: $f'(x) = \dfrac{(x^2+x+1)(1)-(x+1)(2x+1)}{(x^2+x+1)^2}$

$$= \frac{x^2+x+\cancel{1}-2x^2-3x-\cancel{1}}{(x^2+x+1)^2} = \frac{-x^2-2x}{(x^2+x+1)^2}$$

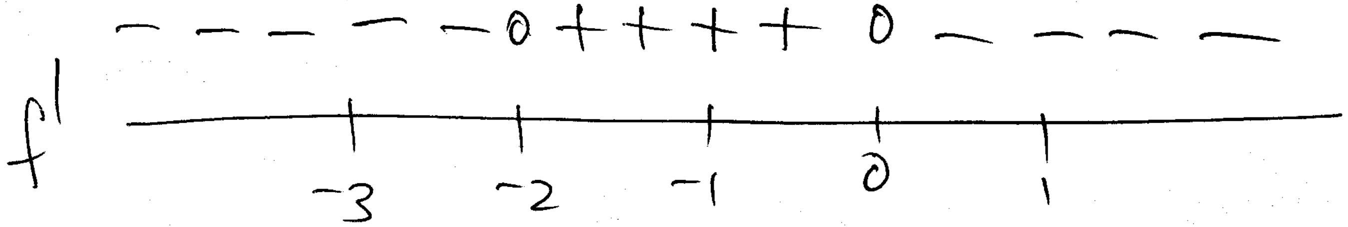
$-x^2-2x=0$

$-x(x+2)=0$

$x=0 \quad x=-2 \leftarrow$ crit #s.

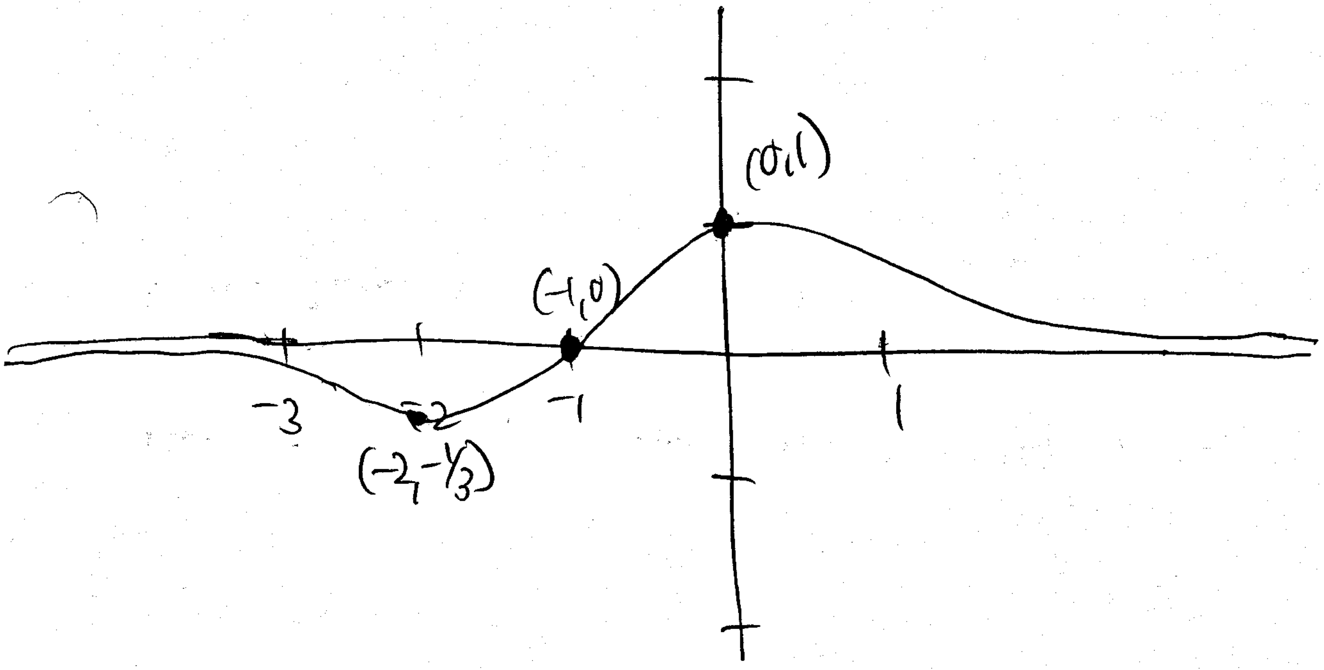
$f(0)=1 \quad f(-2) = \dfrac{-1}{4-2+1} = \dfrac{-1}{3}$

$(0,1) \quad (-2,-\frac{1}{3}) \leftarrow$ critical points.



$$f'(-3) = \frac{-9+6}{(\text{—})^2} < 0 \qquad f'(-1) = \frac{-1+2}{(\quad)^2} > 0$$

$$f'(1) = \frac{-1-2}{(\quad)^2} < 0$$



$$\lim_{x \to \pm\infty} \frac{x+1}{x^2+x+1} = \lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{x} = 0 \quad \checkmark$$

3.2. Concavity and Points of Inflection

Definition

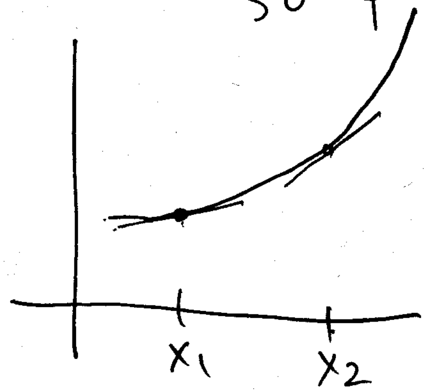
If $f(x)$ is differentiable on the interval $a < x < b$, then the graph of f is

- concave upward on $a < x < b$ if f' is increasing on the interval
- concave downward on $a < x < b$ if f' is decreasing on the interval

What does f'' tell us?

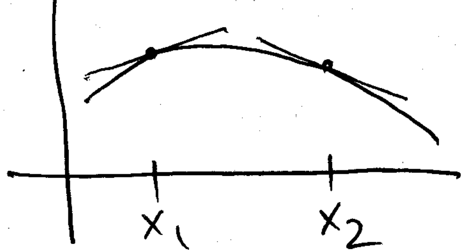
f'' is the derivative of f',

so f'' tells us if f' is increasing or decreasing.



f' increasing

concave up.



f' decreasing

concave down

Concavity

Second Derivative Procedure for Determining Intervals of Concavity

Step 1. Find all values of x for which $f''(x) = 0$ or $f''(x)$ does not exist, and mark these numbers on a number line. This divides the line into a number of open intervals.

Step 2. Choose a test number c from each interval determined in step 1 and evaluate f''. Then

- If $f''(c) > 0$, the graph of $f(x)$ is concave upward on $a < x < b$.
- If $f''(c) < 0$, the graph of $f(x)$ is concave downward on $a < x < b$.

Concavity

Example

Determine intervals of concavity for the function

$$f(x) = 3x^5 - 10x^4 + 11x - 17$$

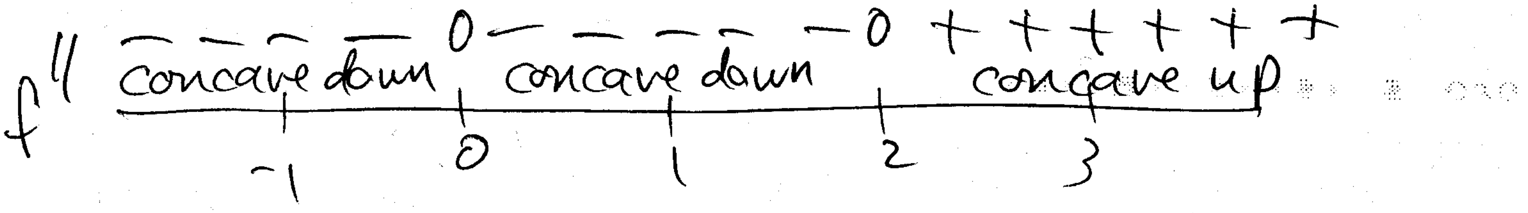
$$f'(x) = 15x^4 - 40x^3 + 11$$

$$f''(x) = 60x^3 - 120x^2$$

$$60x^3 - 120x^2 = 0$$

$$60x^2(x-2) = 0$$

$x = 0 \qquad x = 2$



$f''(-1) = 60(-3) < 0 \qquad f''(1) = 60(-1) < 0$

$f''(3) = 60 \cdot 9 \cdot (1) > 0$

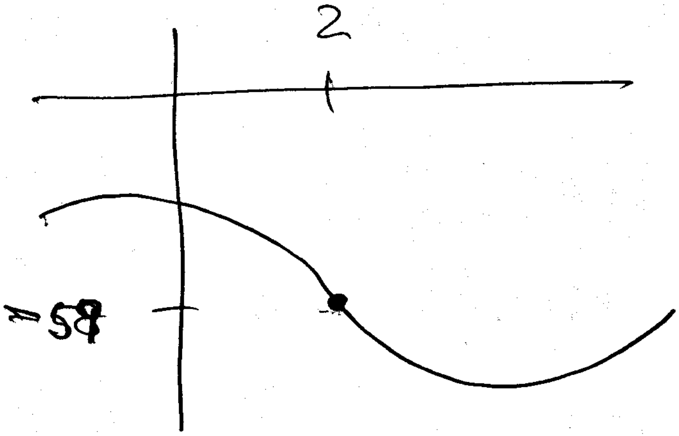
$$\boxed{\begin{array}{l}\text{Conc. down on } (-\infty,0)\cup(0,2)\\ \text{Conc up on } (2,\infty)\end{array}}$$

$f(x) = 3x^5 - 10x^4 + 11x - 17$

has an <u>inflection point</u> at $x = 2$.

Inflection point: $(2, -58)$

$f(2) = 96 - 160 + 22 - 17 = 118 - 177 = -59$



Inflection Points

Definition

An inflection point is a point $(c, f(c))$ on the graph of f where the concavity changes.

At such a point, either $f''(c) = 0$ or $f''(c)$ does not exist.

Procedure for finding the Inflection Points

Step 1. Compute $f''(x)$ and determine all points in the domain of f where either $f''(c) = 0$ or $f''(c)$ does not exist.

Step 2. For each number c found in step 1, determine the sign of f'' to the left of $x = c$ and to the right of $x = c$. If $f''(x) > 0$ on one side and $f''(x) < 0$ on the other side, then $(c, f(c))$ is an inflection point for f.

Curve Sketching with the Second Derivative

Example

Determine where the function

$$f(x) = x^3 + 3x^2 + 1$$

is increasing and decreasing, and where its graph is concave up and concave down. Find all relative extrema and points of inflection, and sketch the graph.

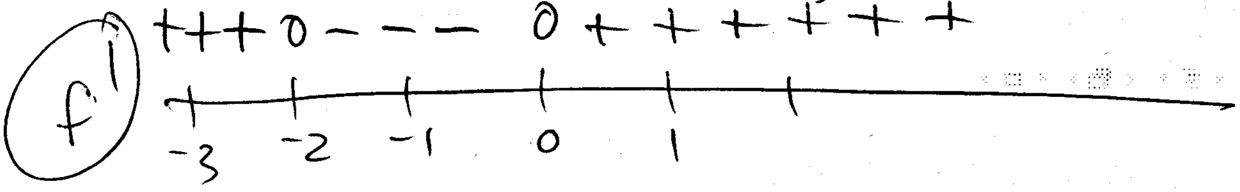
$f'(x) = 3x^2 + 6x$

$3x^2 + 6x = 0$

$3x(x+2) = 0$

$x = 0 \quad x = -2$ ← crit #s

Crit. points: $(0,1)$ $(-2,5)$

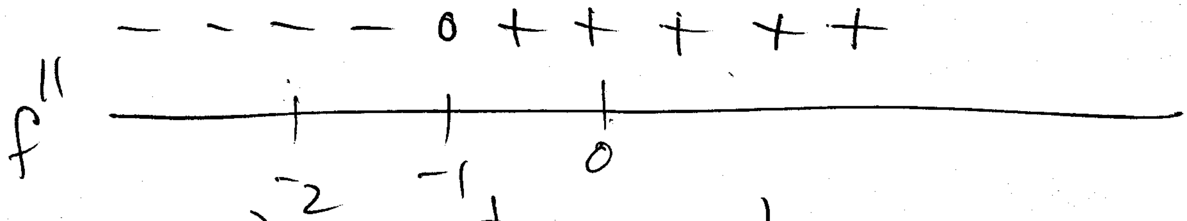


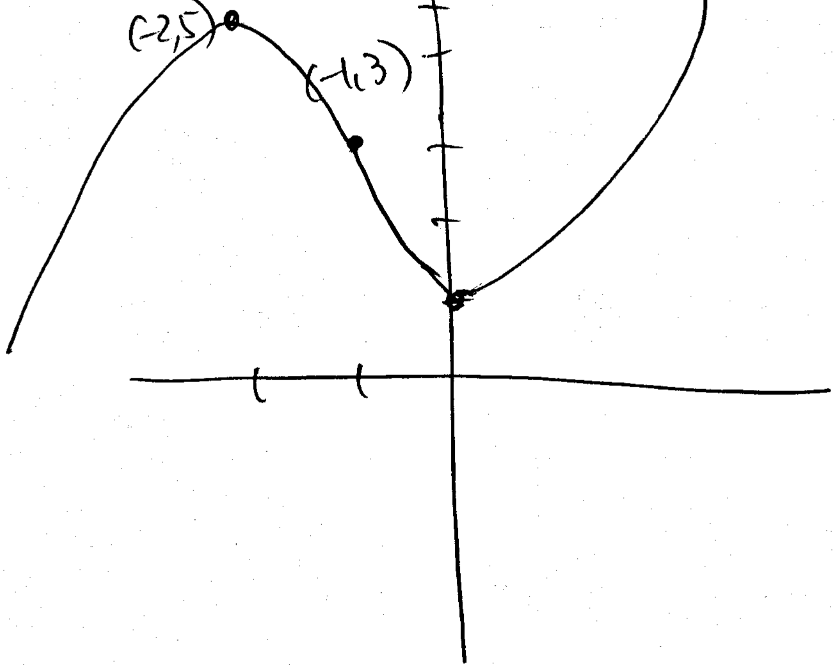
$f''(x) = 6x + 6$

$6x + 6 = 0$

$x = -1$ ← possible inflection

$(-1,3)$ ← inflection point


− − − − 0 + + + + +
f''
-2
-1
0


(-2,5)
(-1,3)

Curve Sketching with the Second Derivative

Example

Determine where the function

$$f(x) = \frac{x^2}{x^2 + 3}$$

is increasing and decreasing, and where its graph is concave up and concave down. Find all relative extrema and points of inflection, and sketch the graph.

Concavity and Inflection Points

Example

The first derivative of a certain function $f(x)$ is

$$f'(x) = x^2 - 2x - 8.$$

(a) Find intervals on which f is increasing and decreasing.

(b) Find intervals on which the graph of f is concave up and concave down.

(c) Find the x coordinate of the relative extrema and inflection points of f.

The Second Derivative Test

Suppose $f''(x)$ exists on an open interval containing $x = c$ and that $f'(c) = 0$.

- If $f''(c) > 0$, then f has a relative minimum at $x = c$.
- If $f''(c) < 0$, then f has a relative maximum at $x = c$.

However, if $f''(c) = 0$ or if $f''(c)$ does not exist, the test is inconclusive and f may have a relative maximum, a relative minimum, or no relative extremum at all at $x = c$.

The Second Derivative Test

Example

Find the critical points of

$$f(x) = x^3 + 3x^2 + 1$$

and use the second derivative test to classify each critical point as a relative maximum or minimum.