Lecture 4: Completion of a Metric Space

Closure vs. Completeness.

Recall the statement of Lemma **??**(b): *A subspace $\mathcal{M}$ of a metric space $\mathcal{X}$ is closed if and only if every convergent sequence $\{x_n\} \subseteq \mathcal{X}$ satisfying $\{x_n\} \subseteq \mathcal{M}$ converges to an element of $\mathcal{M}$.*

Also recal the statement of Lemma **??**: *A closed subspace of a complete metric space is complete.*

Lemma **??** sounds a lot like the definition of completeness. Note however that it does not say that a closed subspace is complete. As Lemma **??** shows, a closed subspace of a *complete* metric space is complete. It may not be true that a closed subspace of an incomplete space is complete.

For example, any space is a closed subspace *of itself* (this is a result of the tautology that a sequence of elements converging to an element of the space converges to an element of the space). We have seen examples of spaces that are not complete. More specifically, the space $(\mathbf{Q}, |\cdot|)$ is closed as a subspace of itself but is not closed as a subspace of $(\mathbf{R}, |\cdot|)$. This is because there are sequences in $\mathbf{Q}$ which converge to elements of $\mathbf{R}\setminus\mathbf{Q}$, that is, they converge to irrational numbers.

The point is that the notion of closure is always applied to a *subspace* of some larger space. A subspace may be closed as a subspace of one space but not as the subspace of another. Completeness, on the other hand, is a property of a metric space regardless of whether or not it is considered as a subspace of a larger space.

Question: Can every metric space be made complete? That is, can the "holes" in an incomplete metric space be always be filled in? The answer is yes. The new space is referred to as the *completion* of the space. The procedure is as follows. Given an incomplete metric space M, we must somehow define a larger complete space in which M sits. Then the closure of M in this larger space is defined to be its completion.

Completion of a Metric Space.

Example. We have seen that the metric space $(\mathbf{Q}, |\cdot|)$ is not complete. That is, there are Cauchy sequences in $\mathbf{Q}$ which do not converge to an element of $\mathbf{Q}$. However, we know that the real numbers $\mathbf{R}$ have the property that: (1) $\mathbf{Q} \subseteq \mathbf{R}$, (2) $\mathbf{R}$ is complete, (3) $\mathbf{Q}$ is dense in $\mathbf{R}$, that is, any real number can be written as the limit of a sequence of rational numbers. So what we really want to do is add to $\mathbf{Q}$ all of its "limit points." For that we need to define a *larger space* in which $\mathbf{Q}$ sits and which contains its limit points.

How do we construct $\mathbf{R}$ from $\mathbf{Q}$? That is, given that we know the rational numbers, how do we define the real numbers? There are several approaches.

1. Dedekind cuts. The approach takes advantage of ordering properties of $\mathbf{Q}$. We will not discuss it in this course.

2. Cauchy sequences. This approach takes advantage of the topological or metric properties of **Q**.

Consider for example our series development for $\pi/4$. We know that the sequence of partial sums of the series $\{s_N\}_{N=0}^{\infty}$ is Cauchy but does not converge in **Q**.

We wish to fill in the "hole" in **Q** to which it "converges." So we say that the "hole" is represented by the Cauchy sequence, that is, we say

$$\frac{\pi}{4} \sim \{s_0,\, s_1,\, s_2,\, \ldots\}.$$

There are several problems with this approach that we must solve. For example, if we say that $\frac{\pi}{4}$ is equal to the Cauchy sequence $\{s_N\}_{N=0}^{\infty}$ then we must also say that it is equal to any other Cauchy sequence that converges to $\pi/4$. For example, $\{s_N\}_{N=5}^{\infty}$ or $\{s_N + 1/N\}_{N+1}^{\infty}$, or even $\{3/4,\, 31/40,\, 314/400,\, 3141/4000,\, \ldots\}$ all converge to $\frac{\pi}{4}$. So the first step is to actually identify $\frac{\pi}{4}$ with the collection of all Cauchy sequences that converge to it. This raises another problem: How do we know if two Cauchy sequences converge to the same "hole" in **Q**?

There are other problems also: (1) Since **Q** is supposed to be a subset of **R**, can we represent rational numbers as collections of Cauchy sequences as well? (2) How do we define the distance between two real numbers? Can we do it in such a way that it is the same as the usual distance? (3) Are the real numbers defined in this way actually a complete space?

Theorem 2 below gives a general construction based on the approach of filling in holes in an incomplete space by identifying the hole with a class of Cauchy sequences converging to that hole. The issues listed above are dealt with one at a time, making for a rather tedious proof. However, all details should be verified by the reader.

Definition 1. *Let $(\mathcal{X}, d)$ be a metric space. A subset $B \subseteq \mathcal{X}$ is said to be* **dense** *in $\mathcal{X}$ if each element $x \in \mathcal{X}$ can be written as the limit of a sequence in B. That is, given $x \in \mathcal{X}$, there exists a sequence $\{b_n\}_{n=1}^{\infty} \subseteq B$ such that $b_n \to x$ as $n \to \infty$ in $\mathcal{X}$.*

Theorem 2. *Let $\mathcal{X} = (\mathcal{X}, d)$ be an incomplete metric space. Then there exists a metric space $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}, \tilde{d})$ which is complete and such that $\mathcal{X}$ is sometric to a dense subset of $\tilde{\mathcal{X}}$.*

Proof: Consider the collection of all Cauchy sequences in $\mathcal{X}$ and define the following relation between such sequences: $\{x_n\} \sim \{y_n\}$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

CLAIM 1. *The above defined relation is an equivalence relation on the set of all Cauchy sequences in $\mathcal{X}$.*

Define the set $\tilde{\mathcal{X}}$ to be the set of all equivalence classes of Cauchy sequences in $\mathcal{X}$. Denote an element of $\tilde{\mathcal{X}}$ by $\tilde{x} = [\{x_n\}]$, where $[\{x_n\}]$ denotes the equivalence class containing $\{x_n\}$. Define the metric $\tilde{d}$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = d([\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty} d(x_n, y_n).$$

We must show that this definition makes sense.
Claim 2. *The limit defining $\tilde{d}$ exists.*
Proof of Claim 2: Let $\{x_n\}$ and $\{y_n\}$ be Cauchy in $\mathcal{X}$. We will show that the sequence $\{d(x_n, y_n)\}$ is a Cauchy sequence of real numbers. By the triangle inequality,

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &\le |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \\ &\le d(y_n, y_m) + d(x_n, x_m). \end{aligned}$$

Since each of the sequences $\{x_n\}$ and $\{y_n\}$ is Cauchy, it is clear that for n, m large enough, the right side of the above can be made arbitrarily small. Claim 2 is proved.
Claim 3. *The definition of $\tilde{d}$ is independent of the choice of representative, that is, if $\{x_n^1\} \sim \{x_n^2\}$ and $\{y_n^1\} \sim \{y_n^2\}$ then $\lim_{n\to\infty} d(x_n^1, y_n^1) = \lim_{n\to\infty} d(x_n^2, y_n^2)$.*
Proof of Claim 3: By the triangle inequality,

$$\begin{aligned} |d(x_n^1, y_n^1) - d(x_n^2, y_n^2)| &\le |d(x_n^1, y_n^1) - d(x_n^1, y_n^2)| + |d(x_n^1, y_n^2) - d(x_n^2, y_n^2)| \\ &\le d(y_n^1, y_n^2) + d(x_n^1, x_n^2). \end{aligned}$$

The right side goes to zero by the definition of the equivalence relation. Claim 3 is proved.
Claim 4. *$\tilde{d}$ defines a metric on the set $\tilde{\mathcal{X}}$.*
Claim 5. *$\tilde{\mathcal{X}}$ is a complete metric space.*
Proof of Claim 5: Let $\{\tilde{x}_k\}_{k=1}^{\infty}$ be Cauchy in $\tilde{\mathcal{X}}$ with $\tilde{x}_k = [\{x_n^k\}]$ for each k. $\{\tilde{x}_k\}$ Cauchy means that given $\epsilon > 0$, there exists N such that if k, $j \ge N$, then

$$\tilde{d}(\tilde{x}_k, \tilde{x}_j) = \lim_{n\to\infty} d(x_n^k, x_n^j) < \epsilon.$$

We must show that $\{\tilde{x}_k\}$ in fact converges. The first step is to define a candidate, $\tilde{x}$, for the limit of $\{\tilde{x}_k\}$. For this we will use a modification of the type of diagonal argument we have already seen.

Define $\tilde{x}_1 = x_1^1$. Since $\{x_n^2\}$ is Cauchy, choose an integer $N(2) > 1$ such that $d(x_{N(2)}^2, x_m^2) < 1/2$ whenever $m \ge N(2)$. Continue in this fashion, choosing for each integer k an integer $N(k)$ such that (1) N(k)¿N(k-1), and (2) $d(x_{N(k)}^k, x_m^k) < 1/k$ whenever $m \ge N(k)$. Define $\tilde{x} = [\{x_{N(n)}^n\}]$.

We must first verify that $\{x_{N(n)}^n\}$ is Cauchy so that $\tilde{x}$ is actually an equivalence class of Cauchy sequences and hence is in $\tilde{\mathcal{X}}$. To do this, we must verify that

$$\lim_{n,m\to\infty} d(x_{N(n)}^n, x_{N(m)}^m) = 0.$$

To this end, note that for any j,

$$d(x^n_{N(n)}, x^m_{N(m)}) \leq d(x^n_{N(n)}, x^n_j) + d(x^n_j, x^m_j) + d(x^m_j, x^m_{N(m)}).$$

Letting $j \to \infty$ on both sides (and noting that the left side does not depend on j) we have that

$$\begin{aligned} d(x^n_{N(n)}, x^m_{N(m)}) &\leq \limsup(d(x^n_{N(n)}, x^n_j) + d(x^n_j, x^m_j) + d(x^m_j, x^m_{N(m)})) \\ &\leq 1/n + 1/m. \end{aligned}$$

Letting now n, $m \to \infty$ gives us the result.

Finally, we must show that $\tilde{x}_k \to \tilde{x}$. Recall that $\tilde{d}(\tilde{x}_k, \tilde{x}) = \lim_{n\to\infty} d(x^k_n, x^n_{N(n)})$ Let $\epsilon > 0$ and choose K so large that $1/K < \epsilon/2$ and if n, $m \geq K$ then $d(x^n_{N(n)}, x^m_{N(m)}) < \epsilon/2$. Now, if $k \geq K$, then

$$\begin{aligned} \tilde{d}(\tilde{x}_k, \tilde{x}) &= \lim_{n\to\infty} d(x^k_n, x^n_{N(n)}) \\ &\leq \limsup_{n\to\infty} d(x^k_n, x^k_{N(k)}) + \limsup_{n\to\infty} d(x^k_{N(k)}, x^n_{N(n)}) \\ &\leq 1/k + \epsilon/2 \leq 1/K + \epsilon/2 < \epsilon. \end{aligned}$$

Hence, $\tilde{M}$ is a complete metric space.

Claim 6. *$\mathcal{X}$ is isometric to a dense subset of $\tilde{\mathcal{X}}$.*

Outline of proof of Claim 6: Define the function $h: \mathcal{X} \longrightarrow \tilde{\mathcal{X}}$ by $h(x) = [\{x_n\}]$ where $x_n = x$ for all n. It is easy to verify that for x, $y \in \mathcal{X}$, $\tilde{d}(h(x), h(y)) = d(x, y)$. To see that $h(\mathcal{X})$ is dense in $\tilde{\mathcal{X}}$, let $s = [\{y_n\}] \in \tilde{M}$. For each k, define $s_k = h(y_k) = [\{x^k_n\}]$ where $x^k_n = y_k$ for all n. It is easy to verify that $\lim_{k\to\infty} s_k = s$.

The significance of Claim 6 is that it gives us a precise way of making the statement that the incomplete metric space $\mathcal{X}$ is actually a subspace of its completion $\tilde{\mathcal{X}}$. That is, we can with confidence think of $\tilde{\mathcal{X}}$ as being identical with $\mathcal{X}$ except that the "holes" in M have been filled in.

Contraction Mappings and Fixed Points.

Definition 3. *Let $(\mathcal{X}, d)$ be as metric space, and let $A: \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping from $\mathcal{X}$ to itself. The point $x \in \mathcal{X}$ is a* **fixed point** *for A if $Ax = x$. The mapping A is a* **contraction mapping** *on $\mathcal{X}$ if there exists a constant $\alpha < 1$ such that $d(Ax, Ay) \leq \alpha d(x, y)$ for all x, $y \in \mathcal{X}$.*

Theorem 4. *Let $(\mathcal{X}, d)$ be a complete metric space and let A be a contraction mapping on $\mathcal{X}$. Then A has a unique fixed point.*

Proof: Let x_0 be an arbitrary point in $\mathcal{X}$. Define the sequence $\{x_n\}_{n=1}^\infty$ by $x_n = Ax_{n-1}$. Since A is a contraction mapping, there is a constant $\alpha < 1$ such that $d(x_n, x_{n+1}) \leq \alpha\, d(x_{n-1}, x_n)$ for all n. A simple induction argument shows that in fact $d(x_n, x_{n+1}) \leq \alpha^n\, d(x_0, x_1)$. Since $\alpha < 1$,

$\sum_{n=1}^{\infty} \alpha^n < \infty$. Therefore, $\{x_n\}$ is Cauchy (see Exercise 12) and since $\mathcal{X}$ is complete, converges to some $x \in \mathcal{X}$. Since A is continuous,

$$Ax = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x,$$

so that x is a fixed point for A.

To see that x is unique note that if y were also a fixed point, then

$$d(x, y) = d(Ax, Ay) \leq \alpha\, d(x, y),$$

which implies that $d(x, y) = 0$ or $x = y$. □

The above theorem gives us not only a proof of the existence of a fixed point, but a scheme for finding the fixed point via the sequence $\{x_n\}$. This scheme is called generally **successive iteration** or **successive approximation**. This is a useful starting point for numerical schemes for solving certain types of equations.

The theorem also gives us an estimate for the error in our iteration scheme. That is, we can estimate how far we are from the solution at each step and halt our numerical algorithm accordingly.

Theorem 5. *Given a contraction mapping A on a complete metric space $(\mathcal{X}, d)$, and an arbitrary point $x_0 \in \mathcal{X}$, define the sequence $\{x_n\}$ as in the proof of Theorem 4. Then*

$$d(x_n, x) \leq \alpha^n \, \frac{d(x_0, x_1)}{1-\alpha}.$$

Proof: Fix n and let $m > n$. Since for any n, $d(x_n, x_{n+1}) \leq \alpha^n\, d(x_0, x_1)$,

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_m) + d(x_m, x) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) + d(x_m, x) \\ &\leq d(x_0, x_1)\left(\sum_{j=n}^{m-1} \alpha^j\right) + d(x_m, x). \end{aligned}$$

Letting now $m \to \infty$ on the right side, we have that

$$\begin{aligned} d(x_n, x) &\leq d(x_0, x_1) \lim_{m\to\infty}\left(\sum_{j=n}^{m-1} \alpha^j\right) + \lim_{m\to\infty} d(x_m, x) \\ &= d(x_0, x_1) \sum_{j=n}^{\infty} \alpha^j \\ &= \alpha^n \, \frac{d(x_0, x_1)}{1-\alpha}. \end{aligned}$$

□