Lecture 3: Compactness.

Definitions and Basic Properties.

Definition 1. *An* **open cover** *of a metric space $\mathcal{X}$ is a collection (countable or uncountable) of open sets $\{\mathcal{U}_\alpha\}$ such that $\mathcal{X} \subseteq \cup_\alpha \mathcal{U}_\alpha$. A metric space $\mathcal{X}$ is* **compact** *if every open cover of $\mathcal{X}$ has a finite subcover. Specifically, if $\{\mathcal{U}_\alpha\}$ is an open cover of $\mathcal{X}$, then there is a finite set $\{\alpha_1, \ldots, \alpha_N\}$ such that $\mathcal{X} \subseteq \cup_{n=1}^N \mathcal{U}_{\alpha_n}$.*

A collection of subsets $\{A_\alpha\}$ of a metric space $\mathcal{X}$ has the **finite intersection property (FIP)** *if for every finite set of indices, $\{\alpha_1, \ldots, \alpha_N\}$, the intersection $\cap_{n=1}^N A_{\alpha_n}$ is non-empty.*

Theorem 2. *(Theorem 1, p. 93, Kolomogorov) A metric space $\mathcal{X}$ is compact if and only if every collection of closed sets with the FIP has nonempty intersection.*

Proof: ($\Longrightarrow$) Suppose that $\mathcal{X}$ is compact and that $\{A_\alpha\}$ has the FIP and has empty intersection. Consider the collection of open sets $\{(A_\alpha)^c\}$ consisting of the complements of the A_α. Since $\cap_\alpha A_\alpha = \emptyset$ then $\cup_\alpha (A_\alpha)^c = \mathcal{X}$ so that $\{(A_\alpha)^c\}$ is an open cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, there exists a finite set $\{\alpha_1, \ldots, \alpha_N\}$ such that $\cup_{n=1}^N (A_{\alpha_n})^c = \mathcal{X}$. But this means that $\cap_{n=1}^N A_{\alpha_n} = \emptyset$ which contradicts the assumption that $\{A_\alpha\}$ has the FIP.

($\Longleftarrow$) Let $\{\mathcal{U}_\alpha\}$ be an open cover of $\mathcal{X}$. Then $\cap_\alpha (\mathcal{U}_\alpha)^c = \emptyset$. Therefore the collection of closed sets $\{(\mathcal{U}_\alpha)^c\}$ cannot have the FIP. Therefore there is a finite collection $\{\alpha_1, \ldots, \alpha_N\}$ such that $\cap_{n=1}^N (\mathcal{U}_{\alpha_n})^c = \emptyset$. But this means that $\cap_{n=1}^N \mathcal{U}_{\alpha_n} = \mathcal{X}$ and $\{\mathcal{U}_\alpha\}$ has a finite subcover. □

Theorem 3.

(a) *(Theorem 2, p. 93, K) Every closed subset of a compact metric space is compact.*

(b) *(Theorem 3, p. 93, K) If $\mathcal{K}$ is a compact subset of a metric space $\mathcal{X}$, then $\mathcal{K}$ is closed*

Proof: (a) Let $\mathcal{M}$ be a closed subspace of the compact space $\mathcal{X}$, and let $\{\mathcal{U}_\alpha\}$ be an open cover of $\mathcal{M}$, since $\mathcal{M}$ is closed, the collection $\{\mathcal{U}_\alpha\} \cup \{\mathcal{M}^c\}$ is an open cover of $\mathcal{X}$ which contains a finite subcover since $\mathcal{X}$ is compact. Therefore, there is a collection $\{\alpha_1, \ldots, \alpha_N\}$ such that $\{\mathcal{U}_{\alpha_n}\}_{n=1}^N \cup \{\mathcal{M}^c\}$ covers $\mathcal{X}$. Hence $\{\mathcal{U}_{\alpha_n}\}_{n=1}^N$ must cover $\mathcal{M}$. Therefore, $\mathcal{M}$ is compact.

(b) Let $\mathcal{K}$ be a compact subset of $\mathcal{X}$ and let $y \notin \mathcal{K}$. It will suffice to show that $y \notin [\mathcal{K}]$ as this would imply that $[\mathcal{K}] \subseteq K$. For each $x \in \mathcal{K}$, let $\epsilon_x = d(x, y)/2$. Then $B(x; \epsilon_x) \cap B(y; \epsilon_x) = \emptyset$. The collection $\{B(x; \epsilon_x)\}_{x \in \mathcal{K}}$ forms an open cover of $\mathcal{K}$ and since $\mathcal{K}$ is compact there is a finite set $\{x_1, \ldots, x_N\}$ such that $\mathcal{K} \subseteq \cup_{n=1}^N B(x; \epsilon_x)$. Let $\epsilon = \min\{\epsilon_{x_1}, \ldots, \epsilon_{x_N}\}$.

Then $B(y;\epsilon) \cap \mathcal{K} = \emptyset$ since if $x \in \mathcal{K}$ then $x \in B(x_n; \epsilon_{x_n})$ for some n, and hence is not in $B(y; \epsilon_{x_n})$ for that n. Since $B(y;\epsilon) \subseteq B(y; \epsilon_{x_n})$, x cannot be in $B(y;\epsilon)$ either. Therefore $y \notin [\mathcal{K}]$. $\square$

Continuous Functions on Compact Spaces

Lemma 4. *A function f from a metric space $\mathcal{X}$ into a metric space $\mathcal{Y}$ is continuous at each point of $\mathcal{X}$ if and only if for every open set $\mathcal{U} \subseteq \mathcal{Y}$, $f^{-1}(\mathcal{U})$ is open.*

Proof: ($\Longrightarrow$) Suppose that f is continuous on $\mathcal{X}$, that $\mathcal{U} \subseteq \mathcal{Y}$ is open, and let $x \in f^{-1}(\mathcal{U})$. Since $f(x) \in \mathcal{U}$ there is an $\epsilon > 0$ such that $B(f(x);\epsilon) \subseteq \mathcal{U}$ and since f is continuous at x a $\delta > 0$ such that if $y \in B(x;\delta)$ then $f(y) \in B(f(x);\epsilon) \subseteq \mathcal{U}$. Therefore, $B(x;\delta) \subseteq f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{U})$ is open.

($\Longleftarrow$) Let $x \in \mathcal{X}$, $\epsilon > 0$, and assume that the inverse image of under f of an open set is open. Then $f^{-1}(B(f(x);\epsilon))$ is open and contains x. Hence there is a $\delta > 0$ such that $B(x;\delta) \subseteq f^{-1}(B(f(x);\epsilon))$. But this is precisely the definition of continuity at x. $\square$

Theorem 5.

(a) *(Theorem 5, p. 94, K) The continuous image of a compact space is compact.*

(b) *(Theorem 6, p. 94, K) A continuous injection of a compact space $\mathcal{X}$ onto a metric space $\mathcal{Y}$ (which also must be compact) is a homeomorphism, that is, the inverse function is also continuous.*

Proof: (a) Let $\mathcal{X}$ be a compact space and f a continuous function from $\mathcal{X}$ into a metric space $\mathcal{Y}$. Let $\{\mathcal{U}_\alpha\}$ be an open cover of $f(\mathcal{X})$. Since f is continuous and by Lemma 4, each of the sets $f^{-1}(\mathcal{U}_\alpha)$ is open and the collection $\{f^{-1}(\mathcal{U}_\alpha)\}$ forms an open cover of $\mathcal{X}$ (for given $x \in \mathcal{X}$, there is an α_x such that $f(x) \in \mathcal{U}_{\alpha_x}$, and hence $x \in f^{-1}(\mathcal{U}_{\alpha_x})$). Since $\mathcal{X}$ is compact there is a finite set $\{\alpha_1, \ldots, \alpha_N\}$ such that $\mathcal{X} \subseteq \cup_{n=1}^{N} f^{-1}(\mathcal{U}_{\alpha_n})$. But this means that $f(\mathcal{X}) \subseteq \cup_{n=1}^{N} \mathcal{U}_{\alpha_n}$, and so $f(\mathcal{X})$ is compact.

(b) Since we are trying to show that f^{-1} is continuous, it is enough to show that $(f^{-1})^{-1} = f$ maps open sets to open sets. Let $\mathcal{U} \subseteq \mathcal{X}$ be open. Then its complement $\mathcal{U}^c$ is closed and hence compact since $\mathcal{X}$ is compact by Theorem 3(a). By part (a), the $f(\mathcal{U}^c)$ is also compact and by Theorem 3(b) it is closed. Since $f(\mathcal{U}^c) = f(\mathcal{U})^c$ (Hint: This is because f is injective and onto) $f(\mathcal{U})^c$ is also closed, hence $f(\mathcal{U})$ is open. $\square$

Compactness and Completeness

Theorem 6. *(Theorem 7, p. 94, K) If a metric space $\mathcal{X}$ is compact then every infinite subset of $\mathcal{X}$ has a limit point.*

Proof: Suppose $\mathcal{X}$ is compact and let $\mathcal{M}$ be an infinite subset of $\mathcal{X}$. We can extract from $\mathcal{M}$ a sequence of distinct points $\{x_n\}_{n=1}^{\infty}$. Let $A_n = \{x_n, x_{n+1}, \ldots\}$ Then $\{[A_n]\}$ is a sequence of closed sets with the FIP. Since $\mathcal{X}$ is compact, there is an $x \in \cap_{n=1}^{\infty} A_n$.

To see that x is a limit point of $\mathcal{M}$, let $\epsilon > 0$ and consider $B(x;\epsilon)$. Since $x \in [A_n]$ for all n, and since A_n is closed, x is a closure point for each A_n. If x were an isolated point for some A_n, then x would equal x_m for some m. In this case, x could not be an isolated point for any A_n with $n > m$ since all of the x_n are distinct. This means that for all sufficiently large n, $B(x;\epsilon) \cap \{x_n, x_{n+1}, \ldots\} \neq \emptyset$. Hence $B(x;\epsilon) \cap \mathcal{M} \setminus \{x\} \neq \emptyset$ and x is a limit point of $\mathcal{M}$. $\square$

Definition 7. *A subset $\mathcal{M}$ of a metric space $\mathcal{X}$ is* **totally bounded** *if for every $\epsilon > 0$ there is a finite set of points $\{x_n\}_{n=1}^{N}$ such that $\mathcal{M} \subseteq \cup_{n=1}^{N} B(x_n;\epsilon)$. Another way of putting this is that every point of $\mathcal{M}$ is within a distance ϵ of at least one of the x_n. The set $\{x_n\}$ is called a* **finite** ϵ**–net** *of $\mathcal{M}$.*

Remark 8. *(a) A subset $\mathcal{M}$ of a metric space $\mathcal{X}$ is bounded if for some $x \in \mathcal{X}$ and $r > 0$, $\mathcal{M} \subseteq B(x;r)$.*

(b) It is always true that a totally bounded set is bounded. (Exercise)

Example 9. (a) Every bounded set in $\mathbf{R}^n$ is totally bounded. To see this, let $\delta > 0$ and consider the lattice of points $\delta\mathbf{Z}^n$. Then every point of $\mathbf{R}^n$ is within a distance of at most $\delta\sqrt{n}/2$ from a point of $\delta\mathbf{Z}^n$. Therefore, the collection of balls $\{B(x;\delta\sqrt{n}/2)\}_{x\in\delta\mathbf{Z}^n}$ covers all of $\mathbf{R}^n$. For each $R > 0$, let $B_R = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \colon |x_i| \leq R\}$. That is, B_R is a box centered at the origin whose sides are of length $2R$. Note that for any $R > 0$, only finitely many balls from the set $\{B(x;\delta\sqrt{n}/2)\}_{x\in\delta\mathbf{Z}^n}$ are required to cover B_R. Specifically, if $R < N\delta$ for some $N \in \mathbf{N}$ then at most N^n balls are required to cover B_R.

Now, if $\mathcal{M} \subseteq \mathbf{R}^n$ is bounded and $\epsilon > 0$ is given, choose δ so small that $\delta\sqrt{n}/2 < \epsilon$ and $N \in \mathbf{N}$ so large that $\mathcal{M}$ is contained in the box $B_{N\delta}$. Then $\mathcal{M}$ can be covered by finitely many balls of radius ϵ.

(b) The unit sphere in ℓ^2 is bounded but not totally bounded. This implies that total boundedness and boundedness are not equivalent properties in infinite dimensional spaces.

To see why this is true, note that by definition the unit sphere is bounded. Now let e_n be the sequence in ℓ^2 consisting of all zeros except for a 1 in the n^{th} position. Then all of the e_n are on the unit sphere and if $n \neq m$, $d(e_n, e_m) = \sqrt{2}$. Therefore, if $\epsilon < \sqrt{2}/2$, then no two e_n can sit in the same ϵ ball and we will require infinitely many of them to cover the unit sphere, which therefore cannot be totally bounded.

Theorem 10. *Let $\mathcal{X}$ be a totally bounded metric space. Then the following are equivalent.*

(a) *$\mathcal{X}$ is compact.*

(b) *Every* **countable** *open cover of $\mathcal{X}$ admits a finite subcover.*

(c) *Every* **countable** *collection of closed sets with the FIP has nonempty intersection.*

(d) *Every infinite subset of $\mathcal{X}$ has a limit point.*

Proof: (a)$\Longrightarrow$(b) Follows from the definition of compactness.

(b)$\Longrightarrow$(a) Let $\{\mathcal{U}_\alpha\}$ be an open cover (countable or uncountable) of $\mathcal{X}$. Since $\mathcal{X}$ is totally bounded, we can define a doubly infinite sequence of points in $\mathcal{X}$ as follows. For each $k = 1, 2, 3, \ldots$ let $\{x_{n,k}\}_{n=1}^{N_k}$ be a finite collection with the property that $\mathcal{X} \subseteq \cup_{n=1}^{N_k} B(x_{n,k}, 1/k)$. For each $x \in \mathcal{X}$, choose n and k so that $x \in B(x_{n,k}, 1/k)$ and $B(x_{n,k}, 1/k) \subseteq \mathcal{U}_\alpha$ for some α. This can be done by noting that $x \in \mathcal{U}_\alpha$ for some α so that for some $\epsilon > 0$, $B(x;\epsilon) \subseteq \mathcal{U}_\alpha$. Now choose k so large that $1/k < \epsilon/2$ and let n be such that $x \in B(x_{n,k}, 1/k)$. Then if $y \in B(x_{n,k}, 1/k)$, $d(x, y) \leq d(x, x_{n,k}) + d(x_{n,k}, y) < 1/k + 1/k < \epsilon$. Hence $B(x_{n,k}, 1/k) \subseteq \mathcal{U}_\alpha$. The collection of all such $x_{n,k}$ chosen this way is a countable set and the associated $\mathcal{U}_\alpha$ must be countable as well. Since each $x \in \mathcal{X}$ is accounted for by some $x_{n,k}$, this countable collection of open sets covers $\mathcal{X}$.

Hence by (b) this countable subcover of $\mathcal{X}$ admits a finite subcover. Hence $\mathcal{X}$ is compact.

(b)$\Longleftrightarrow$(c) This is the same argument as in Theorem 2.

(a)$\Longrightarrow$(d) This is Theorem 6.

(d)$\Longrightarrow$(c) Let $\{\mathcal{F}_n\}_{n=1}^\infty$ be a countable collection of closed sets satisfying the FIP. Define $A_N = \cap_{n=1}^N \mathcal{F}_n$. Then the sequence $\{A_N\}_{N+1}^\infty$ satisfies (1) A_N is nonempty and closed for each N, (2) $A_{N+1} \subseteq A_N$ for all N, (3) $\{A_N\}_{N+1}^\infty$ has the FIP, and (4) $\cap_{n=1}^\infty \mathcal{F}_n = \cap_{N+1}^\infty A_N$.

There are now two possibilities: First, for some N_0, $A_{N_0} = A_{N_0+1} = \cdots$ in which case $\cap_{N=1}^\infty A_N = A_{N_0}$ and (c) is satisfied. The other possibility is that the A_N are distinct for infinitely many N. By renumbering the $\{A_N\}$ if necessary, we can assume that the A_N are distinct for all N. In this case, we can choose an infinite sequence of distinct points $\{x_n\}$ such that $x_N \in A_N$ for all N. By (d), this sequence must have a limit point, call it x. Since the sequence $\{x_N, x_{N+1}, \ldots\} \subseteq A_N$ for each N, x is a limit point for each set A_N. Since each A_N is closed, $x \in A_N$ for each N. Hence $x \in \cap_{N+1}^\infty A_N$ and by property (4) above, $\cap_{n=1}^\infty \mathcal{F}_n$ is not empty. Therefore (c) holds in this case also. $\square$

Theorem 11. *(Theorem 2, p. 100, K) A metric space $\mathcal{X}$ is compact if and only if it is totally bounded and complete.*

Proof: ($\Longrightarrow$) Suppose $\mathcal{X}$ is compact. Given $\epsilon > 0$, we can cover $\mathcal{X}$ with the collection of open balls $\{B(x;\epsilon)\}_{x\in\mathcal{X}}$ and extract a finite subcover of the form $\{B(x_n;\epsilon)\}_{n=1}^N$. Hence $\mathcal{X}$ is totally bounded. Now suppose that $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$. We can assume that $\{x_n\}$ is an infinite set because any Cauchy sequence with only finitely many distinct elements must have all of the x_n the same for all n sufficiently large. By Theorem 6, x_n has a limit point and hence $\{x_n\}$ must converge to that limit point and $\mathcal{X}$ is complete.

($\Longleftarrow$) Suppose that $\mathcal{X}$ is totally bounded and complete. To show it is compact, we will use Theorem 10 and show that every infinite subset of $\mathcal{X}$ has a limit point. To this end, assume that $\mathcal{M}$ is an infinite subset of $\mathcal{X}$ and extract from $\mathcal{M}$ a sequence $\{x_n\}_{n=1}^\infty$ of distinct points.

We will define a subsequence of this sequence as follows. Since $\mathcal{X}$ is totally bounded, there is a finite collection of balls of radius 1 covering $\mathcal{X}$. At least one of these balls must contain infinitely many of the x_n. Choosing such a ball allows us to define a subsequence $\{x_{1,n}\}_{n=1}^\infty$ contained in one such ball. Using total boundedness again, we know that $\mathcal{X}$ is covered by finitely many balls of radius 1/2, at least one of which contains infinitely many of the $x_{1,n}$. Choosing one such ball allows us to extract a subsequence of $\{x_{1,n}\}$ which we will call $\{x_{2,n}\}_{n=1}^\infty$. Continuing in this fashion, we have a doubly infinite sequence of points $\{x_{k,n}\}$ with the property that (1) $\{x_{k+1,n}\}_{n=1}^\infty \subseteq \{x_{k,n}\}_{n=1}^\infty$, (2) $d(x_{k,n}, x_{k,m}) < 1/k$ for all n, $m \in \mathbf{N}$. For each k, choose an element of the sequence $\{x_{k,n}\}$ which has not previously been chosen. In this way, define a sequence of distinct points $\{y_k\}_{k=1}^\infty$ with the property that $y_k \in \{x_{k,n}\}_{n=1}^\infty$.

Since for every $m > 0$, $d(y_n, y_{n+m}) < 2/n$, $\{y_k\}$ is a Cauchy sequence in $\mathcal{X}$ and hence converges (say to y) by the completeness of $\mathcal{X}$. To see that y is a limit point of $\mathcal{M}$, note that for every $\epsilon > 0$, $B(y;\epsilon)$ contains infinitely many of the y_k and hence infinitely many points in $\mathcal{M}$. $\square$