Lecture 2: Topology of Metric Spaces

Definition 1. *Let $(\mathcal{X}, d)$ be a metric space and let $\mathcal{M} \subseteq \mathcal{X}$. Then the pair $(\mathcal{M}, d)$ is a metric space in its own right and is called a* **subspace** *of $(\mathcal{X}, d)$.*

Definition 2. *Let $(\mathcal{X}, d)$ be a metric space. The* open ball *of radius $r > 0$ about $x \in \mathcal{X}$ is the set $B(x; r) = \{y \colon d(x, y) < r\}$. The* closed ball *is the set $B(x; r) = \{y \colon d(x, y) \le r\}$.*

A subset $\mathcal{U} \subseteq \mathcal{X}$ is **open** *if for each $x \in \mathcal{U}$, there is an $\epsilon > 0$ such that $B(x; \epsilon) \subseteq \mathcal{U}$. A set $\mathcal{F} \subseteq \mathcal{X}$ is* **closed** *if its complement F^c is open.*

Given $x \in \mathcal{X}$, a set $\mathcal{N} \subseteq \mathcal{X}$ is a **neighborhood** *of x if there is an open ball $B(x; r) \subseteq \mathcal{N}$.*

Theorem 3. *(Theorem 3, p. 49, Theorem 5, p. 50, Kolmogorov)*

(a) *Let $\{\mathcal{U}_\alpha\}$ be a system (finite, countable or uncountable) of open sets in a metric space X. Then $\cup_\alpha \mathcal{U}_\alpha$ is open.*

(b) *Let $\{\mathcal{U}_n\}_{n=1}^N$ be a finite system of open sets in $\mathcal{X}$. Then $\cap_{n=1}^N \mathcal{U}_n$ is open.*

Proof: (a) Let $x \in \cup_\alpha \mathcal{U}_\alpha$, then for some fixed α_0, $x \in \mathcal{U}_{\alpha_0}$. Since $\mathcal{U}_{\alpha_0}$ is open, there exists $\epsilon > 0$ such that $B(x; \epsilon) \subseteq \mathcal{U}_{\alpha_0}$. Thus also, $B(x; \epsilon) \subseteq \cup_\alpha \mathcal{U}_\alpha$ and $\cup_\alpha \mathcal{U}_\alpha$ is open.

(b) Let $x \in \cap_{n=1}^N \mathcal{U}_n$. Then $x \in \mathcal{U}_n$ for all n and for each n there is an $\epsilon_n > 0$ such that $B(x; \epsilon_n) \subseteq \mathcal{U}_n$. Let ϵ be the smallest of the ϵ_n. Then $B(x; \epsilon) \subseteq B(x; \epsilon_n) \subseteq \mathcal{U}_n$ for all n and hence $B(x; \epsilon) \subseteq \cap_{n=1}^N \mathcal{U}_n$. Therefore $\cap_{n=1}^N \mathcal{U}_n$ is open. □

Corollary 4.

(a) *Let $\{\mathcal{F}_\alpha\}$ be a system (finite, countable or uncountable) of closed sets in a metric space X. Then $\cap_\alpha \mathcal{F}_\alpha$ is closed.*

(b) *Let $\{\mathcal{F}_n\}_{n=1}^N$ be a finite system of closed sets in $\mathcal{X}$. Then $\cup_{n=1}^N \mathcal{F}_n$ is closed.*

Example 5. (a) The set $(0, 1) \subseteq \mathbf{R}$ is open in $\mathbf{R}$ with the usual absolute value metric. The set $[0, 1]$ is closed in $\mathbf{R}$ because its complement, the union of two open intervals, is an open set. The sets $\emptyset$ and $\mathbf{R}$ are both open and closed as subsets of $\mathbf{R}$. The set $(0, 1]$ is neither open nor closed in $\mathbf{R}$.

(b) Any singleton (that is, a set consisting of only one point) in $\mathbf{R}$ is closed. The set $[0, 1] \cup \{2\}$ is closed in $\mathbf{R}$ as it is the union of two closed sets.

(c) Any subset $\mathcal{M}$ of a metric space $(\mathcal{X}, d)$ where d is the discrete topology is closed.

(d) Let $\mathcal{F}_n = [1/n, 1 - 1/n]$. Then each $\mathcal{F}_n$ is closed but $\cup_{n=1}^\infty \mathcal{F}_n = (0, 1)$ which is not closed.

(e) Let $\mathcal{U}_n = (-1/n, 1+1/n)$. Then each $\mathcal{U}_n$ is open but $\cap_{n=1}^{\infty}\mathcal{U}_n = [0,1]$ which is not open.

Definition 6. *The point $x \in \mathcal{X}$ is a* closure point *of $\mathcal{M} \subseteq \mathcal{X}$ if for every $\epsilon > 0$ $B(x;\epsilon)$ contains a point of $\mathcal{M}$. (Kolmogorov calls such a point a* contact point.*)*

The point $x \in \mathcal{X}$ is a limit point *or an* accumulation point *of $\mathcal{M}$ if for every $\epsilon > 0$, $B(x;\epsilon)$ contains infinitely many points of $\mathcal{M}$. Equivalently, $x \in \mathcal{X}$ is a limit point of $\mathcal{M}$ if for all $\epsilon > 0$, $B(x:\epsilon)$ contains a point of $\mathcal{M} \setminus \{x\}$.*

A closure point of $\mathcal{M}$ which is not a limit point of $\mathcal{M}$ is called an isolated point.

Lemma 7.

(a) *A point x is a closure point of $\mathcal{M}$ if and only if there is a sequence of points $\{x_n\}$ of $\mathcal{M}$ converging to x.*

(b) *A point x is a limit point of $\mathcal{M}$ if and only if there is a sequence of distinct points $\{x_n\}$ of $\mathcal{M}$ converging to x.*

(c) *An isolated point of $\mathcal{M}$ must be in $\mathcal{M}$.*

Proof: (a) For each n, let $\epsilon = 1/n$ and choose $x_n \in B(x;1/n) \cap \mathcal{M}$. Then $x_n \in \mathcal{M}$ for each n and $x_n \to x$.

(b) Use the same procedure as in (a) to choose the x_n but since x is an accumulation point, we can guarantee that x_n is not equal to x nor to any of the (finitely many) previously chosen points.

(c) Suppose that x is an isolated point of $\mathcal{M}$ which is not in $\mathcal{M}$. Let $\epsilon_1 = 1$ and let x_1 be any point in $B(x;\epsilon_1) \cap \mathcal{M}$. Note that $x_1 \neq x$. Let $\epsilon_2 = \min\{1/2, d(x,x_1)\}$ and choose x_2 to be any point in $B(x;\epsilon_2) \cap \mathcal{M}$. Continue in this fashion, letting $\epsilon_n = \min\{1/n, d(x,x_{n-1})\}$, and choosing x_n to be any point in $B(x;\epsilon_n) \cap \mathcal{M}$. Since $\epsilon_n \leq 1/n$, $x_n \to x$, and since $d(x,x_n) < d(x,x_{n-1})$, x_n is not equal to any of the previously chosen x_m. Thus $\{x_n\}$ is a sequence of distinct points in $\mathcal{M}$ converging to x. By definition then x is a limit point of $\mathcal{M}$ and hence cannot be an isolated point. □

Definition 8. *The* **closure** *of $\mathcal{M}$ is the set of all closure points of $\mathcal{M}$, and is denoted $[\mathcal{M}]$. Note that the closure of $\mathcal{M}$ consists of $\mathcal{M}$ together with all of its limit points. A subspace $(\mathcal{M}, d)$ is* **closed** *provided that $\mathcal{M}$ is closed.*

Example 9. (a) The set $\mathcal{M} = (0,1) \subseteq \mathbf{R}$ is not closed in $\mathbf{R}$ because 1 is a closure point of $\mathcal{M}$ but is not in $\mathcal{M}$.

(b) The point 2 is a closure point of the set $\mathcal{M} = [0,1] \cup \{2\}$. In fact it is an isolated point of $\mathcal{M}$.

(c) Any subset $\mathcal{M}$ of a metric space $(\mathcal{X}, d)$ where d is the discrete topology is closed.

Lemma 10. *Let $\mathcal{M}$ be a subspace of a metric space $\mathcal{X}$. Then the following are equivalent.*

(a) $\mathcal{M}$ *is* closed.

(b) $[\mathcal{M}] = \mathcal{M}$.

(c) *Every convergent sequence $\{x_n\} \subseteq \mathcal{X}$ satisfying $\{x_n\} \subseteq \mathcal{M}$ converges to an element of $\mathcal{M}$.*

Proof: (a)$\Longleftrightarrow$(b). Assume that $\mathcal{M}$ is closed. Since $[\mathcal{M}]$ consists of $\mathcal{M}$ together with its limit points, always $\mathcal{M} \subseteq [\mathcal{M}]$. Suppose $x \notin \mathcal{M}$. Since $\mathcal{M}$ is closed its complement is open and since x is in that complement there is an $\epsilon > 0$ such that $B(x; \epsilon) \cap \mathcal{M} = \emptyset$. But by definition, this means that x is not a closure point of $\mathcal{M}$ so that $x \notin [\mathcal{M}]$.

Now assume that $[\mathcal{M}] = \mathcal{M}$. It will be sufficient to show that $[\mathcal{M}]$ is closed. If $x \notin [\mathcal{M}]$ then x is not a closure point of $\mathcal{M}$ so that there is an $\epsilon > 0$ such that $B(x; \epsilon) \cap \mathcal{M} = \emptyset$. But since $\mathcal{M} = [\mathcal{M}]$, $B(x; \epsilon) \cap [\mathcal{M}] = \emptyset$ also. Therefore the complement of $[M]$ is open and $[M]$ is closed.

(b)$\Longleftrightarrow$(c). Assume that $[\mathcal{M}] = \mathcal{M}$ and that and let $\{x_n\}$ be a sequence in $\mathcal{M}$ converging to the point x. By the definition of convergence, for every $\epsilon > 0$ there is at least one of the x_n in $B(x; \epsilon)$. Hence x is a closure point of $\mathcal{M}$ and so is in $[\mathcal{M}]$. But since $[\mathcal{M}] = \mathcal{M}$, $x \in \mathcal{M}$.

Now suppose that (b) does not hold and that there is a point $x \in [\mathcal{M}]$ which is not in $\mathcal{M}$. But by Lemma 7(a) there is a sequence $\{x_n\}$ in $\mathcal{M}$ converging to x. But this means that (c) also does not hold. $\square$

Theorem 11. *(Properties of Closure.)*

(a) *If $\mathcal{M} \subseteq \mathcal{N}$ then $[\mathcal{M}] \subseteq [\mathcal{N}]$.*

(b) $[[\mathcal{M}]] = [\mathcal{M}]$.

(c) $[\mathcal{M} \cup \mathcal{N}] = [\mathcal{M}] \cup [\mathcal{N}]$.

Proof: (a) Let $x \in [\mathcal{M}]$. Then for every $\epsilon > 0$, $B(x; \epsilon) \cap \mathcal{M} \neq \emptyset$. Since $\mathcal{M} \subseteq \mathcal{N}$, $B(x; \epsilon) \cap \mathcal{N} \neq \emptyset$ also. Hence $x \in [\mathcal{N}]$.

(b) This follows immediately from Lemma 10 and the fact that $[\mathcal{M}]$ is closed.

(c) By part (a), and since both $\mathcal{M}$ and $\mathcal{N}$ are subsets of $\mathcal{M} \cup \mathcal{N}$, $[\mathcal{M}] \cup [\mathcal{N}] \subseteq [\mathcal{M} \cup \mathcal{N}]$. By definition of closure, $\mathcal{M} \subseteq [\mathcal{M}]$ and $\mathcal{N} \subseteq [\mathcal{N}]$, so that $\mathcal{M} \cup \mathcal{N} \subseteq [\mathcal{M}] \cup [\mathcal{N}]$. By part (a), $[\mathcal{M} \cup \mathcal{N}] \subseteq [[\mathcal{M}] \cup [\mathcal{N}]] = [\mathcal{M}] \cup [\mathcal{N}]$ by Lemma 10 and the fact that the union of two closed sets is closed. $\square$

Lemma 12. *A closed subspace of a complete metric space is complete.*

Proof: Let $\mathcal{M}$ be a subspace of the complete metric space $\mathcal{X}$, and let $\{x_n\}$ be a Cauchy sequence in $\mathcal{M}$. Then also, $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$ and hence converges to some $x \in \mathcal{X}$. Therefore, x is a closure point of $\mathcal{M}$ but since $\mathcal{M}$ is closed, $x \in \mathcal{M}$. Hence $\mathcal{M}$ is complete. $\square$