3.3 Second order Quasi-linear Hyperbolic Equations

Consider

$$a(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x^2} + b(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x \partial y}$$

$$+ c(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial y^2} + e(x,y,u,u_x,u_y) = 0$$

Define

$$p = \frac{\partial u}{\partial x} \qquad q = \frac{\partial u}{\partial y}$$

$$r = \frac{\partial^2 u}{\partial x^2} \qquad s = \frac{\partial^2 u}{\partial x \partial y} \qquad t = \frac{\partial^2 u}{\partial y^2}$$

so

$$ar + bs + ct + e = 0$$

As in the first order case, we'll procede by trying to follow the solution along characteristics.

- Parameterize a curve C_0 by ρ [we used s before but now s is something else...] so points along it are identified by $(x(\rho), y(\rho))$.

Along C_0 we have

$$\frac{dp}{d\rho} = \frac{\partial^2 u}{\partial x^2}\frac{dx}{d\rho} + \frac{\partial^2 u}{\partial x \partial y}\frac{dy}{d\rho} \quad ; \quad \frac{dq}{d\rho} = \frac{\partial^2 u}{\partial x \partial y}\frac{dx}{d\rho} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{d\rho}$$

or

$$\frac{dp}{d\rho} = r\frac{dx}{d\rho} + s\frac{dy}{d\rho}$$

$$\frac{dq}{d\rho} = s\frac{dx}{d\rho} + t\frac{dy}{d\rho}$$

Use these two equations to eliminate r and t from the PDE

$$ar + bs + ct + e = 0$$

That is,

$$r = \frac{\frac{dp}{d\rho} - s\frac{dy}{d\rho}}{\frac{dx}{d\rho}} \qquad t = \frac{\frac{dq}{d\rho} - s\frac{dx}{d\rho}}{\frac{dy}{d\rho}}$$

so

$$a\left(\frac{\frac{dp}{d\rho} - s\frac{dy}{d\rho}}{\frac{dx}{d\rho}}\right) + bs + c\left(\frac{\frac{dq}{d\rho} - s\frac{dx}{d\rho}}{\frac{dy}{d\rho}}\right) + e = 0$$

$$s\left[-a\left(\frac{dy}{d\rho}\right)^2 + b\left(\frac{dx}{d\rho}\right)\left(\frac{dy}{d\rho}\right) - c\left(\frac{dx}{d\rho}\right)^2\right] + a\frac{dp}{d\rho}\frac{dy}{d\rho} + c\frac{dq}{d\rho}\frac{dx}{d\rho}$$
$$+ e\frac{dx}{d\rho}\frac{dy}{d\rho} = 0$$

Now choose C_0 (via an equation for x and y) such that

$$\boxed{a\left(\frac{dy}{d\rho}\right)^2 - b\left(\frac{dx}{d\rho}\right)\left(\frac{dy}{d\rho}\right) + c\left(\frac{dx}{d\rho}\right)^2 = 0} \leftarrow \text{Equation for characteristics}$$

the PDE is then

$$\boxed{a\frac{dp}{d\rho}\frac{dy}{d\rho} + c\frac{dq}{d\rho}\frac{dx}{d\rho} + e\frac{dx}{d\rho}\frac{dy}{d\rho} = 0}$$ along a curve C_0.

also $\frac{du}{d\rho} - p\frac{dx}{ds} + q\frac{dy}{ds}$

To simplify things (and connect with the book) take the parameterization

$$x = x(p) = p \qquad \Rightarrow \quad \frac{dx}{dp} = 1 \quad \frac{dy}{dp} = \frac{dy}{dx}$$
$$y = y(p) = y(x)$$

then we have

$$\boxed{\begin{array}{ll} * & a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0 \\ ** & a\frac{dp}{dx}\frac{dy}{dx} + c\frac{dz}{dx} + e\frac{dy}{dx} = 0 \end{array}}$$

Note: $*$ will have two real roots $\left(\frac{dy}{dx}\right)$ if

$$b^2 - 4ac > 0 \qquad \text{(i.e. from the quadratic formula)}$$

which means two real characteristic curves.

Note also: the original equation

$$au_{xx} + bu_{xy} + cu_{yy} + e = 0$$

We classify this equation based on the properties of the matrix

$$A = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix}$$

- A singular (degenerate) [PARABOLIC]
- A positive or negative definite [ELLIPTIC]
- A indefinite [HYPERBOLIC]

The eigenvalues of this matrix satisfy

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0$$

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$

Since A is symmetric we know the eigenvalues are real. Then,

if $\det A > 0$ both $\lambda_1, \lambda_2 \geq 0$ or < 0 (same sign) (ELLIPTIC)

if $\det A < 0$ one positive one negative eigenvalue (HYPERBOLIC)

So the case for hyperbolic occurs when

$$ac - \tfrac{1}{4}b^2 < 0$$ (see also week 1 notes!)

or $b^2 - 4ac > 0$ (same conditions as above).

So, assuming $b^2-4ac>0$ (hyperbolic case)

let the two real roots of

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

be written

$$\frac{dy}{dx} = f(x,y,u,u_x,u_y)$$

$$\frac{dy}{dx} = g(x,y,u,u_x,u_y)$$

i.e. $f,g = \dfrac{b \pm \sqrt{b^2-4ac}}{2}$

along each of these two characteristic curves the equation

$$a\frac{dp}{dx}\frac{dy}{dx} + c\frac{dq}{dx} + e\frac{dy}{dx} = 0$$

applies. Further,

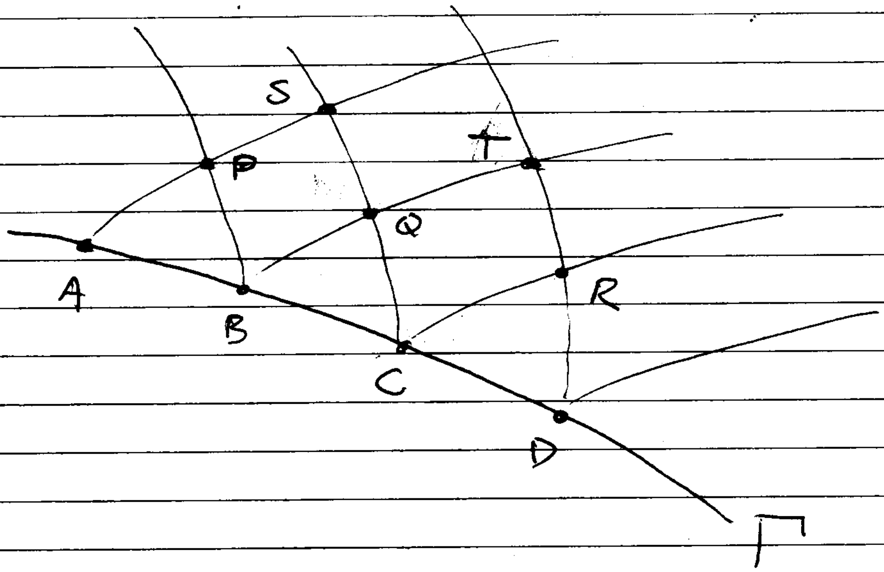
$$\frac{du}{dx} = p + q\frac{dy}{dx}$$

(i.e. $du = p\,dx + q\,dy$

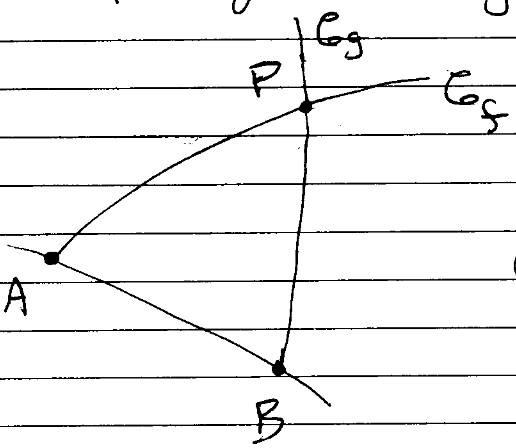
or $\dfrac{du}{ds} = p\dfrac{dx}{ds} + q\dfrac{dy}{ds}$)

Method of Characteristics: Solution procedure.

Assume initial info (u, p, q) given on Γ (Γ not a characteristic).



examine steps required to go from A, B to P.



at P we need

x_P, y_P, q_P, p_P, u_P

These steps are then applied everywhere to obtain solutions elsewhere on the "grid."
Note: we solve for the grid location...

The two characteristic curves are defined by

$$\frac{dy}{dx} = f \quad \text{and} \quad \frac{dy}{dx} = g$$

so take

$$\boxed{(y_P - y_A) = f_A (x_P - x_A)} \quad \text{AND} \quad \boxed{(y_P - y_B) = g_B (x_P - x_B)}$$

Solve these two equations for two unknowns x_P, y_P

$$\begin{bmatrix} f_A & -1 \\ g_B & -1 \end{bmatrix} \begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x_A f_A - y_A \\ x_B g_B - y_B \end{bmatrix}$$

~~x_P ... $(x_B g_B - y_B)$~~

$$x_P = \frac{(y_A - x_A f_A) + (x_B g_B - y_B)}{g_B - f_A}$$

$$y_P = \frac{f_A(x_B g_B - y_B) - g_B(x_A f_A - y_A)}{g_B - f_A}$$

Next solve

$$a \frac{dp}{dx}\frac{dy}{dx} + c\frac{dq}{dx} + e\frac{dy}{dx} = 0$$

along each of these characteristics

(f): $a\frac{dp}{dx} f + c\frac{dq}{dx} + ef = 0$

OR

$$a(dp)f + c\,dq + e\underbrace{f\,dx}_{dy} = 0$$

(g): $a\frac{dp}{dx} g + c\frac{dq}{dx} + eg = 0$

OR

$$a(dp)g + c\,dq + e\underbrace{g\,dx}_{dy} = 0$$

So

$$a_A f_A (p_P - p_A) + c_A (q_P - q_A) + e_A (y_P - y_A) = 0$$

AND

$$a_B g_B (p_P - p_B) + c_B (q_P - q_B) + e_B (y_P - y_B) = 0$$

solve these equations for p_P and q_P.

$$\begin{bmatrix} a_A f_A & c_A \\ a_B g_B & c_B \end{bmatrix} \begin{bmatrix} p_P \\ q_P \end{bmatrix} = \begin{bmatrix} a_A f_A p_A + c_A q_A - e_A (y_P - y_A) \\ a_B g_B p_B + c_B q_B - e_B (y_P - y_B) \end{bmatrix}$$

$\Rightarrow$ p_P and q_P

Finally, to get u_P use $du = p\,dx + q\,dy$

$$u_P - u_A = \frac{(p_A + p_P)}{2}(x_P - x_A) + \frac{(q_A + q_P)}{2}(y_P - y_A)$$

OR

$$u_P - u_B = \frac{(p_B + p_P)}{2}(x_P - x_B) + \frac{(q_B + q_P)}{2}(y_P - y_B)$$

Once x_P, y_P, p_P, q_P and u_P are found, we can go back and re-use equations for x_P, y_P, p_P, q_P with average coefficients as in previous work on 1st order equations (see text p. 82, 83)

EXAMPLE:

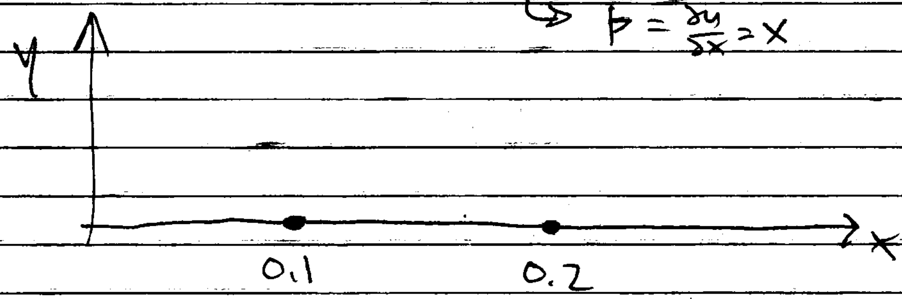
solve $\dfrac{\partial^2 u}{\partial x^2} - (x+y)^2 \dfrac{\partial^2 u}{\partial y^2} = 0$

$$\begin{cases} a = 1 \\ b = 0 \\ c = -(x+y)^2 \end{cases}$$

at the first characteristic grid point between $x=0.1$ and $x=0.2$ for $y>0$. Given: $y=0$ for $0 \leq x \leq 1$

$$u = 1 + \tfrac{1}{2}x^2 \qquad q = \frac{\partial u}{\partial y} = 2x$$

$$\hookrightarrow p = \frac{\partial u}{\partial x} = x$$

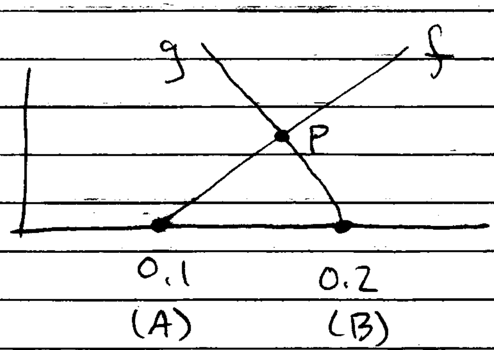


characteristics: $\left(\dfrac{dy}{dx}\right)^2 - (x+y)^2 = 0$

$\hookrightarrow \dfrac{dy}{dx} = x+y = f$ (>0 initially at $y=0$ for $x>0$)

$\dfrac{dy}{dx} = -(x+y) = g$ (<0 initially at $y=0$ for $x>0$

so



Note: $x_A = 0.1$

$y_A = 0$

$f_A = 0.1$

$a_A = 1$

$c_A = -(0.1)^2$

$b_A = e_A = 0$

$p_A = 0.1$

$q_A = 0.2$

$x_B = 0.2$

$y_B = 0$

$g_B = -0.2$

$a_B = 1$

$c_B = -(0.2)^2$

$b_B = e_B = 0$

$p_B = 0.2$

$q_B = 0.4$

see hyper_2nd.m

$\left\{ \begin{array}{l} \text{upq\_hyper2nd.m} \\ \text{abc\_hyper2nd.m} \\ \text{fg\_hyper2nd.m} \end{array} \right.$

$\Rightarrow$ $x_P = 0.1667$

$y_P = 0.0067$

$p_P = 0.1467$

$q_P = 0.6667$

$u_P = 1.0178$ (based on BP path)

$u_P = 1.0161$ (based on AP path)

3.4 Rectangular Nets + Finite Difference Methods for 2nd Order Hyperbolic Equations

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c = \text{wave speed}$$

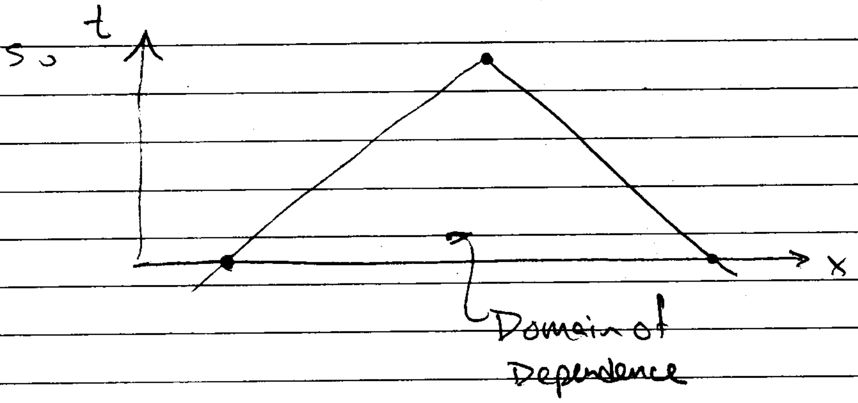
possibly + b.c. $u=\alpha$ $x=0$, $u=\beta$ $x=1$

$u(x,0)=f(x)$, $u_t(x,0)=g(x)$

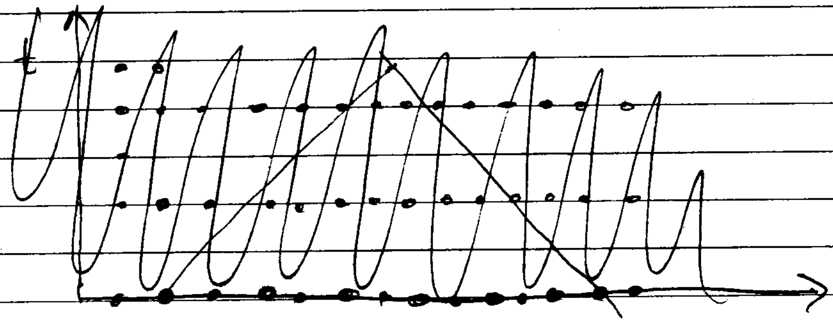
$\Rightarrow$ characteristics: $x+ct=\text{const.}$
$x-ct=\text{const.}$

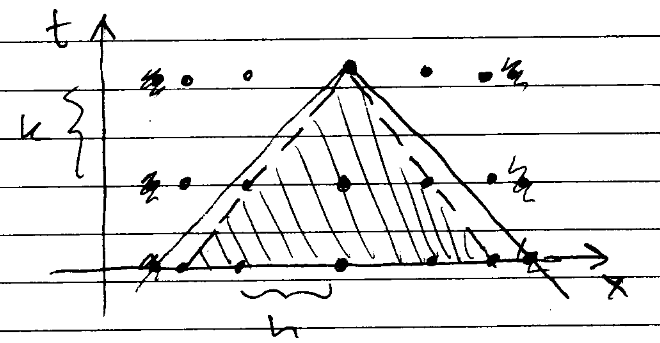
i.e. $\left(\left(\frac{dx}{dt}\right)^2 = c^2 \right)$

So

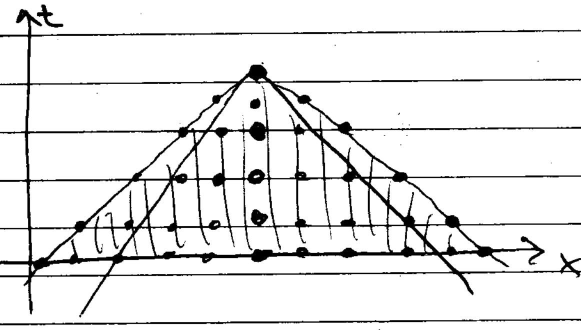


Finite Difference Grid





k too big – UNSTABLE



k small enough – STABLE

Discretize $u_{tt} = c^2 u_{xx}$

2nd order central time + space differencing

$$\frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{k^2} = c^2 \frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{h^2}$$

OR

$$\boxed{u_j^{i+1} = 2u_j^i - u_j^{i-1} + \frac{c^2k^2}{h^2}\left(u_{j+1}^i - 2u_j^i + u_{j-1}^i\right)}$$

Note: • require $\frac{c^2k^2}{h^2} \leq 1$ (CFL condition) for numerical stability

• explicit, but we need information at two previous time levels (i and $i-1$) to advance in time.

• To get started ($t=0 \Rightarrow i=1$)

$$u_j^1 = f(x_j)$$

From $\frac{\partial u}{\partial t} = g(x)$ at $t=0$ $\Rightarrow$ $\frac{u_j^2 - u_j^0}{2k} = g(x_j)$

so $u_j^0 = u_j^2 - 2kg(x_j)$ ← use this for u_j^{i-1} when $i=2$