1.8 Countability of the Rationals Numbers

Key Points:
- definition of a countable set
- $\mathbb{Q}$ is a countable set (example 1.14 ...)
- $\mathbb{R}$ is an uncountable set (Thm 1.8.1 - Cantor)
- Decimal Expansions
- unions of countable sets (Exercises 1.87, 1.88)
- $\mathbb{I}$ (irrational numbers) is uncountable (1.90 exercise)
- closed/open sets + their complements (exercise 1.91)

Definition 1.8.1 (Countable Set)

A set S is called countable if it is an infinite set for which it is possible to arrange all the elements of S into a sequence. That is, S is countable if $S = \{s_1, s_2, s_3, \ldots, s_k, \ldots\}$ with each element of S listed exactly once in the sequence.

NOTE: S is uncountable iff S is neither finite nor countable

An alternative way to think of this is that S is countable iff there exists a function $f: \mathbb{N} \to S$ that is both one-to-one and onto S.

"no two distinct n's in $\mathbb{N}$ go to the same s in S"

"f doesn't miss any elements of S"

(see Wade, p. 29)

Recall: (Definition of "one-to-one" + "onto")

Let X and Y be sets and $f : X \to Y$

i) f is said to be <u>one-to-one</u> (<u>injective</u>, or f an injection) if and only if

$$x_1, x_2 \in X \quad \text{and} \quad f(x_1) = f(x_2)$$

$$\text{implies } x_1 = x_2$$

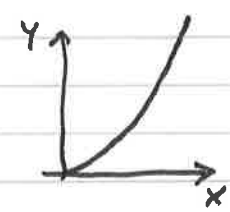
(i.e. $f(x_1) \neq f(x_2)$ unless $x_1 = x_2$)

(ii) f is said to be <u>onto Y</u> (<u>surjective</u>, or f a surjection) if and only if for each $y \in Y$ there is an $x \in X$ such that $y = f(x)$

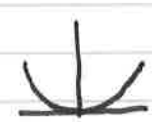
(iii) f is called a <u>bijection</u> if and only if it is both <u>one-to-one</u> and <u>onto Y</u>

<u>EX</u>

- $f(x) = x^2$ is one-to-one from $[0, \infty)$ ($x \geq 0$) and onto $[0, \infty)$.



- $f(x) = x^2$ is not one-to-one for $-1 \leq x \leq 1$ (but it is onto $[0, 1]$).



EXAMPLE

Let E denote the set of odd integers. ($E \subset \mathbb{N}$ and $E \neq \mathbb{N}$).

$$\begin{array}{cccccccccc} E: & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & \dots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \end{array}$$

so $f(n) = 2n-1$ provides the map from $\mathbb{N}$ to E that is both one-to-one and onto E.

Therefore, the set of odd integers in countable.

Comments:

- S is a finite set iff either $S = \emptyset$ (empty set) or there exists ~~an~~ one-to-one function that maps $\{1,2,3,\dots,n\}$ onto S for some $n \in \mathbb{N}$.

- S is uncountable iff S is neither finite nor countable.

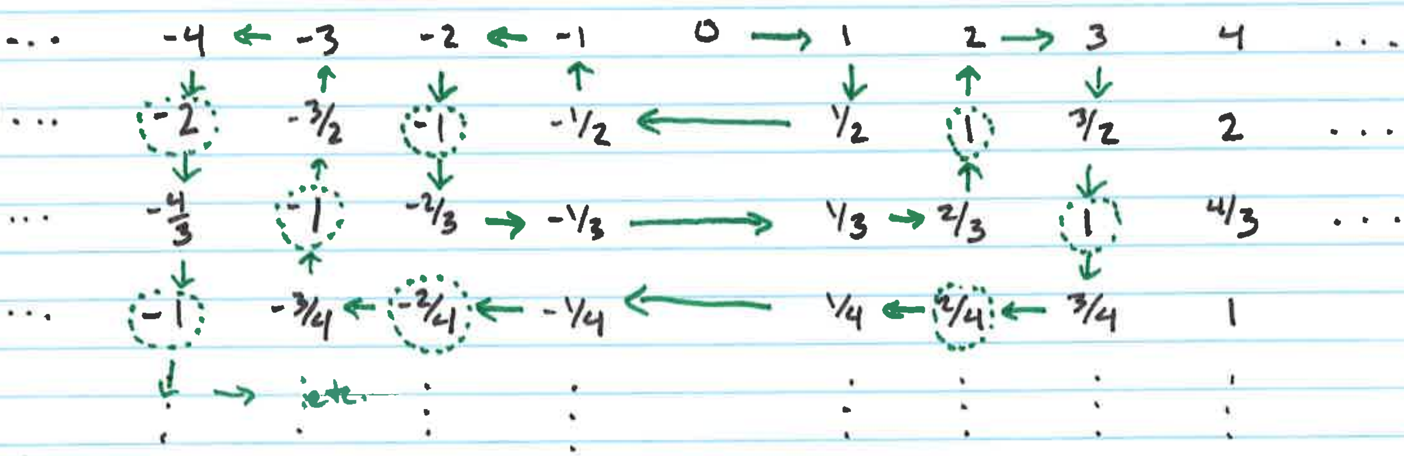
EXAMPLES (of finite, countable + uncountable sets)

- $S = \{2, 4, 6, 8, 10\}$: finite set

- $S = \{2, 4, 6, 8, 10, \ldots\}$ (all even integers) : Countable set

- $\mathbb{Q}$: countable (see discussion / proof in Richardson page 32, example 1.14)

Idea: (see Ross, p. 50)



- The green path shows "onto" (i.e. we don't miss any #'s in $\mathbb{Q}$.

- We can set up a rule to "skip" values we've already covered so that the map is "one-to-one".

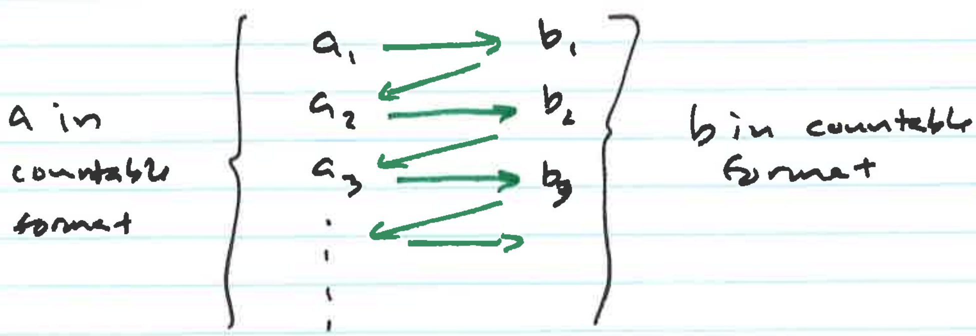
- $\mathbb{R}$ is uncountable (Thm 1.8.1 – Cantor) (see proof in Richardson, p. 34)

- The open interval $(0,1)$ is uncountable

Other Comments

(Exercise 1.87)

$$\boxed{\text{If } A \text{ and } B \text{ are countable, then } A \cup B \text{ is also countable.}}$$



use the above arrangement (skipping repeats) to see that $A \cup B$ is also countable.

(Exercise 1.90)

Observing that $\underset{\text{rationals}}{\mathbb{Q}} \cup \underset{\text{irrationals}}{\mathbb{I}} = \underset{\text{reals}}{\mathbb{R}}$

and $\mathbb{Q}$ is countable while $\mathbb{R}$ is uncountable

we can conclude $\mathbb{I}$ is uncountable — otherwise the union of two countable sets is countable would provide a contradiction that $\mathbb{R}$ is uncountable.

not assigned 2011 Fall semester

HW (1.8) 1.87a, 1.89, 1.90, 1.91

in class | some in class | in class | def of "closed set"