Categorical Properties of Topoi

Let $\mathcal{E} = Sh_J(\mathcal{C}) \underset{i}{\overset{a}{\rightleftarrows}} Set^{\mathcal{C}^{op}}$ be a Grothendieck topos.

$a = (\cdot)^{++} \dashv i$.

<u>Prop</u>: $\mathcal{E}$ is both complete and cocomplete.

- Limits are computed in $Set^{\mathcal{C}^{op}}$
- To compute colimits, first compute them in $Set^{\mathcal{C}^{op}}$ and then apply a.

(This is a general fact about reflective subcategories).

Let $\mathcal{D}$ be a category with pullbacks, and let $f: X \longrightarrow Y \in \mathcal{D}$.

$\rightsquigarrow \quad f^*: \mathcal{D}/Y \longrightarrow \mathcal{D}/X$

$$\begin{tikzcd} Z \arrow[d, "g"] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

<u>Def</u> $\mathcal{D}$ as above is a category in which <u>colimits are universal</u> if for each $f: X \longrightarrow Y$, f^* preserves small colimits.

This basically says that if $F: I \longrightarrow \mathcal{D}$ is a small diagram, and

$$\begin{tikzcd} & \underset{D}{\mathrm{colim}}\, F(D) \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a diagram, then

$$X \times_Y (\underset{D}{\mathrm{colim}}\, F(D)) \cong \underset{D}{\mathrm{colim}}\, (X \times_Y F(D)).$$

<u>Prop</u> Colimits are universal in any (Groth.) topos $\mathcal{E}$.

<u>Pf</u> The fact that colimits are universal in $\mathbf{Set}$ is an easy exercise. Since limits and colimits in $\mathbf{Set}^{\mathcal{C}^{op}}$ are computed "object-wise" it follows that colimits are also universal in $\mathbf{Set}^{\mathcal{C}^{op}}$.

Now suppose
$$\begin{tikzcd} & \varinjlim_D F(D) \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
is a diagram in $\mathcal{E} = Sh_J(\mathcal{C})$ with $F : I \to \mathcal{E}$.

Consider
$$\begin{tikzcd} & \varinjlim_D iF(D) \arrow[d] \\ iX \arrow[r] & iY \end{tikzcd}$$
in $\mathbf{Set}^{\mathcal{C}^{op}}$, with $i : \mathcal{E} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{op}}$, $a \dashv i$.

Then $iX \times_{iY} (\varinjlim_D iF(D)) \cong \varinjlim_D iX \times_{iY} iF(D)$

apply a $\Rightarrow$ $X \times_Y (\varinjlim_D F(D)) \cong \varinjlim_D X \times_Y F(D)$ (since a preserve colimits and pullbacks. ☒

<u>Def</u> Let $\coprod_i A_i$ be a coproduct in a category $\mathcal{D}$ with $(s_j : A_j \longrightarrow \coprod_i A_i)_j$ the associated cocone. This coproduct is said to be <u>disjoint</u> if

1) each s_j is a monomorphism
2) $\forall\, j, j'$ the pullback
$$\begin{tikzcd} A_j \times_{\coprod_i A_i} A_{j'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_{j'} \arrow[d, tail, "s_{j'}"] \\ A_j \arrow[r, tail, "s_j"'] & \coprod_i A_i \end{tikzcd}$$

has $A_j \times_{\coprod_i A_i} A_{j'} = \emptyset$.

Prop In a (Groth) topos, all coproducts are disjoint.

Pf. This is true for Set ($\amalg$ = disjoint union), and $\therefore$ for $\mathrm{Set}^{\mathcal{C}^{op}}$. Since $a = (\cdot)^{++}$ preserves colimits and pullbacks, it follows that 2) holds. For 1) notice that

$S : B \to A$ is mono

$\Updownarrow$

$$\begin{tikzcd} B \arrow[r, "id"] \arrow[d, "id"'] & B \arrow[d, "s"] \\ B \arrow[r, "s"'] & A \end{tikzcd}$$

is a p.b. diagram.

So 1) follows since a is left exact.

Def Let $\mathcal{D}$ be a category and $D \in \mathcal{D}_0$. An equivalence relation is a subobject $R \rightarrowtail D \times D$ s.t. $\forall\ E \in \mathcal{D}_0$ the subset

$$\mathrm{Hom}_{\mathcal{D}}(E, R) \subset \mathrm{Hom}_{\mathcal{D}}(E, D) \times \mathrm{Hom}_{\mathcal{D}}(E, D)$$

is an equivalence relation on the set $\mathrm{Hom}_{\mathcal{D}}(E, D)$.

Given such an eq'l relation $R \rightarrowtail D \times D$, one may consider the induced pair of morphisms $R \rightrightarrows D$.

Def Given an eq'l relation $R \rightarrowtail D \times D$ as above, the quotient object (if it exists) is $D/R := \mathrm{coeq}(R \rightrightarrows D)$.

Example: Let $f : X \to Y \in \mathcal{D}$ and suppose that $\mathcal{D}$ has pullback. Consider $X \times_Y X \rightarrowtail X \times X$. This determines an eq'l relation on X.

$\wr$

Kernel pair $= X \times_Y X \rightrightarrows X$

Given an eq'l relation $R \rightarrowtail D\times D$ with a quotient object $D/R = \varinjlim(R \rightrightarrows D)$ consider the canonical map

$$p : D \longrightarrow D/R$$

and its kernel pair $D\times_{D/R} D \rightarrowtail D\times D$.

__Def__ An equivalence relation $R \rightarrowtail D\times D$ is __effective__ if $R = D\times_{D/R} D$.

__Note__: In Set, every eq'l relation is effective.

__Prop__ In a (Groth.) topos $\mathcal{E}$, every eq'l relation is effective.

__Pf__ Let $\mathcal{E} \underset{i}{\overset{a}{\leftrightarrows}} Set^{\mathcal{C}^{op}}$, and $R \rightarrowtail F\times F \in \mathcal{E}$ an eq'l rel.

Then $iR \rightarrowtail iF\times iF$ is an eq'l rel in $Set^{\mathcal{C}^{op}}$. Since effectivity involves taking colimits and limits, which can be computed object-wise in Set, it follows that all eq'l relations in $Set^{\mathcal{C}^{op}}$ are effective.

Suppose $q : iF \longrightarrow Q = coeq(iR \rightrightarrows iF)$. Then since eq'l relations in $Set^{\mathcal{C}^{op}}$ are effective,

$$\begin{tikzcd} iF\times_Q iF \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & iF \arrow[d, "q"] \\ iF \arrow[r, "q"'] & Q \end{tikzcd} , \qquad \begin{tikzcd} iF\times_Q iF \arrow[rr, tail] \arrow[d, "\varphi\ \wr"'] & & iF\times iF , \\ iR \arrow[urr, tail] & & \end{tikzcd} \quad (\star)$$

Applying a: $aq : F \longrightarrow aQ = coeq(R \rightrightarrows F) = F/R$, and

$$(\star) \Rightarrow \begin{tikzcd} F\times_{aQ} F \arrow[rr, "a\varphi", "\sim"'] \arrow[dr, tail] & & R \arrow[dl, tail] \\ & F\times F & \end{tikzcd} \Rightarrow R \text{ is effective.}$$

Def A category $\mathcal{D}$ with small colimits and pullbacks is **locally κ-presentable**, for a regular cardinal κ if $\exists$ a small subcategory $\mathcal{D}' \xhookrightarrow{\ell} \mathcal{D}$ s.t.

i) $\mathrm{Lan}_\ell \ell = id_{\mathcal{D}}$ ($\mathcal{D}'$ strongly generates $\mathcal{D}$)

↳ means $\forall D$, $D = \varinjlim_{\substack{D' \to D,\\ D' \in \mathcal{D}'}} D'$.

ii) each $G \in \mathcal{D}_0$ is κ-compact:

$$\mathrm{Hom}(G, \cdot): \mathcal{D} \longrightarrow \mathrm{Set}$$

preserves κ-filtered colimits. It is called just locally pres. if $\exists$ κ s.t..

Example Let $\mathcal{D} = \mathrm{Set}^{\mathcal{C}^{op}}$. Then $\mathcal{D}$ is locally κ-presentable $\forall \kappa$: $\mathcal{D}' = \mathcal{C} \xhookrightarrow[y]{} \mathrm{Set}^{\mathcal{C}^{op}}$.

$$\forall\ C \in \mathcal{C},\quad \underbrace{\mathrm{Hom}(y(c), \cdot)}_{\overset{\shortparallel}{ev_C}} : \mathrm{Set}^{\mathcal{C}^{op}} \longrightarrow \mathrm{Set}$$

$$F \longmapsto \mathrm{Hom}(y(c), F) \cong F(c)$$

preserves all colimits.

Prop $\mathcal{E} = Sh_J(\mathcal{C}) \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Set}^{\mathcal{C}^{op}}$ (with $\mathcal{C}$ small) is locally α-presentable, $\forall\ \alpha > |\mathcal{C}_1|$.

① i preserves α-filtered colimits:

Suffices to show that if $F = \varinjlim_I F_k$ is α-filtered, then the colimit can be computed object-wise. Let $\tilde{F} = \varinjlim_I i(F_k)$.

It suffices to show $\tilde{F}$ is a sheaf. Suppose $R \rightarrowtail y(c)$ is a covering sieve and $\beta: R \longrightarrow \tilde{F}$, wts $\exists!\ \tilde{\beta}: y(c) \longrightarrow \tilde{F}$ s.t.

$$\begin{tikzcd} R \arrow[r, tail] \arrow[dr, "\beta"'] & y(c) \arrow[d, "\tilde{\beta}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \tilde{F} \end{tikzcd}$$

6.

$\forall\ D \in \mathcal{C}_0,\quad \beta_D : R(D) \longrightarrow F(D) = \underset{\overrightarrow{I}}{\operatorname{colim}} F_K(D) = \coprod_{K \in I} F_K(D) / \sim$

$f \longmapsto \beta_D(f) = [a_f]$

for $a_f \in F_{i_f}(D)$ for some $i_f \in I$. $\quad (i_f \longrightarrow i \text{ in } I)$

But since I is α-filtered and $\alpha > |\mathcal{C}_1| \Rightarrow \exists\ i \in I$

and $\tilde{a}_f \in F_i(D)\ \forall f \in R(D)$ s.t. $[\tilde{a}_f] = \beta_D(f)\ \forall f$.

Now consider $g : E \longrightarrow D$, and

$$\begin{tikzcd} R(D) \arrow[r, "\beta_D"] \arrow[d, "R(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(D) \arrow[d, "F(g)"] \\ R(E) \arrow[r, "\beta_E"] & F(E) \end{tikzcd}$$

$F_i(g)(\tilde{a}_f) \in F_i(E)$

$\tilde{a}_{f\circ g} \in F_i(E)$

(*)

$\Rightarrow F(g)([a_f]) = F(g)(\beta_D(f)) = \beta_E(R(g)(f)) = [a_{f\circ g}]$ holds in $F(E)$

$\Rightarrow$ it holds in $F_{j_f}(E)$ for some $j_f \in I$, and $|\alpha| > |\mathcal{C}_1| \Rightarrow$

$\exists\ j$ s.t. (*) holds in $F_j\ \forall f$.

$([\hat{a}_f] = [a_f])$

Let $\gamma_D(f) := [\hat{a}_f] \in F_j(D) \rightsquigarrow \gamma : R \longrightarrow iF_j$

s.t.

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[dr, "\beta"'] & iF_j \arrow[d] \\ & F \end{tikzcd}$$

But F_j is a sheaf so $\exists!$

$\tilde{\gamma} : y(C) \longrightarrow iF_j$ s.t.

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[d] & iF_j \\ y(C) \arrow[ur, "\tilde{\gamma}"'] & \end{tikzcd}$$

Let $\tilde{\beta} = y(C) \xrightarrow{\tilde{\gamma}} iF_j \longrightarrow F$. Uniqueness can be shown in a similar way.

② $\forall\ C$, $a(y(C))$ is α-compact in $\mathcal{E}$:

Let $F = \underset{\overrightarrow{I}}{\operatorname{colim}} F_K$ be α-filtered. in $\mathcal{E}$

Then

$$\begin{aligned}
\mathrm{Hom}(a(y(c)), \varinjlim_{I} F_k) &\overset{\dashv}{\cong} \mathrm{Hom}(y(c), i(\varinjlim_{I} F_k)) \\
&\overset{①}{\cong} \mathrm{Hom}(y(c), \varinjlim_{I} i(F_k)) \\
&\cong (\varinjlim_{I} i(F_k)(c)) \\
&\cong \varinjlim_{I} \mathrm{Hom}(y(c), iF_k) \\
&\overset{\dashv}{\cong} \varinjlim_{I} \mathrm{Hom}(ay(c), F_k).
\end{aligned}$$

③ $\forall\, E \in \mathcal{E}$, $\quad iE = \varinjlim_{y(c) \to iE} y(c) \Rightarrow \overset{aiE}{\overset{\wr\wr}{E}} \cong \varinjlim_{y(c) \to iE} ay(c)$

$$\cong \varinjlim_{ay(c) \to E} ay(c).$$

So, if $\mathcal{E}' = \{ay(c),\ c \in \mathcal{C}\} \overset{e}{\hookrightarrow} \mathcal{E}$,

$\mathrm{Lan}_e\, e \cong \mathrm{id}_{\mathcal{E}}$, and each object of $\mathcal{E}'$ is α-compact.

<u>Theorem</u> (Giraud's thm)

A category $\mathcal{E}$ is a Grothendieck topos if and only if:

i) $\mathcal{E}$ is locally presentable
ii) colimits in $\mathcal{E}$ are universal
iii) coproducts in $\mathcal{E}$ are disjoint
iv) every equivalence relation in $\mathcal{E}$ is effective.

Large Topoi

If $\mathcal{C}$ is not small (or not $\mathcal{U}$-small) then

i) may fail but ii) – iv) still hold.

Useful Criterion:

Suppose $\mathcal{E} \simeq Sh_J(\mathcal{C})$, J coming from a Groth. pretopology,

with $\mathcal{C}$ not necc. $\mathcal{U}$-small.

Def $(\mathcal{C}, J)$ is a $\mathcal{U}$-site if $\exists$ a $\mathcal{U}$-small set

$\mathbb{G} = \{G_\alpha \in \mathcal{C}_0\}$ s.t. $\forall$ $C \in \mathcal{C}$ $\exists$ a cover

$(f_{\alpha_j} : G_{\alpha_j} \to C)$ of C. (G_α are called topological generators).

Thm If $(\mathcal{C}, J)$ is a $\mathcal{U}$-site, $Sh_J(\mathcal{C})$ is a topos

(in $\mathcal{U}$).

E.g. $\mathcal{C} = Mfd$, $\mathbb{G} = \{\mathbb{R}^n \mid n \geq 0\}$.

Other Properties of Topoi

Cartesian closure:

Def A category $\mathscr{D}$ is Cartesian-closed if it has binary products, and for each object $D \in \mathscr{D}_0$, the functor

$$\begin{aligned}(\cdot)\times D : \mathscr{D} &\longrightarrow \mathscr{D}\\ E &\longmapsto E\times D\end{aligned}$$

has a right adjoint $\begin{aligned}(\cdot)^D : \mathscr{D} &\longrightarrow \mathscr{D}\\ E &\longmapsto E^D\end{aligned}$

Example 1 $\mathscr{D} = \mathrm{Set}$, $E^D \doteq \mathrm{Hom}(D,E)$:

$\forall$ sets A, $\mathrm{Hom}(A\times D, E) \xrightarrow{\sim} \mathrm{Hom}(A, E^D)$

$$H : A\times D \to E \rightsquigarrow \tilde{H} : A \to E^D$$
$$\tilde{H}(a) : D \to E$$
$$\tilde{H}(a)(d) = H(a,d).$$

Example 2 $\mathscr{D} = \mathrm{Set}^{\mathscr{C}^{op}} \ni F, G.$

$$F^G(C) \overset{\text{Yoneda}}{\cong} \mathrm{Hom}(y(C), F^G) \cong \mathrm{Hom}(y(C)\times G, F)$$

this must define F^G if it exists. Now, suppose $H \in \mathrm{Set}^{\mathscr{C}^{op}}$.

$$\begin{aligned}\mathrm{Hom}(H, F^G) &= \mathrm{Hom}\Big(\varinjlim_{y(C)\to H} y(C), F^G\Big)\\ &\cong \varprojlim_{y(C)\to H} \mathrm{Hom}(y(C)\times G, F)\\ &\cong \mathrm{Hom}\Big(\varinjlim_{y(C)\to H} (y(C)\times G), F\Big)\\ &\overset{\text{colimits universal}}{\cong} \mathrm{Hom}\Big(\big(\varinjlim_{y(C)\to H} y(C)\big)\times G, F\Big)\\ &\cong \mathrm{Hom}(H\times G, F).\end{aligned}$$

Prop If $F \in Sh_J(\mathcal{C})$ and $G \in Set^{\mathcal{C}^{op}}$, then $(iF)^G \in Sh_J(\mathcal{C})$.

Pf Let $R \rightarrowtail y(C) \in Cov_J(C)$. Then $ar : aR \xrightarrow{\sim} ay(C)$.

Suppose
$$\frac{f : R \longrightarrow (iF)^G}{\bar{f} : R \times G \longrightarrow iF}$$

$$\begin{tikzcd}
\bar{f} : R\times G \longrightarrow iF \arrow[r, squiggly, "a"] & a(R\times G) \arrow[r, "a\bar{f}"] \arrow[d, phantom, "\cong" rotate=90] & aiF \cong F \\
& aR \times aG \cong a(y(C)) \times aG \arrow[ur] \arrow[d, phantom, "\cong" rotate=90] & \\
& a(y(C)\times G) &
\end{tikzcd}$$

Thus
$$\frac{\dfrac{a(y(C)\times G) \longrightarrow F}{y(C)\times G \longrightarrow iF}}{y(C) \xrightarrow{\tilde{f}} (iF)^G},$$
s.t.
$$\begin{tikzcd}
R \arrow[r, tail, "r"] \arrow[dr, "f"'] & y(C) \arrow[d, "\tilde{f}"] \arrow[dl, phantom, "\circlearrowright"] \\
& (iF)^G
\end{tikzcd}.$$

Easy to check that $\tilde{f}$ must be unique.

Cor Every Grothendieck topos is Cartesian closed.

Subobject Classifiers

Def A subobject classifier in a category $\mathcal{E}$ is a monomorphism $t : T \rightarrowtail \Omega$ s.t. $\forall$ monomorphisms $m : A \rightarrowtail B$ $\exists!$ $\phi_m : B \longrightarrow \Omega$ s.t.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, tail, "t"] \\
B \arrow[r, "\phi_m"'] & \Omega
\end{tikzcd}$$
is a p.b.

(so t is the "universal subobject")

Example In Set, let $\Omega = \{0,1\}$ 2-element set

and $t : 1 \longrightarrow \Omega$.
$\quad\; * \longmapsto 1$

Given $A \subset B$ $\exists !$ characteristic function

$$\phi_A : B \longrightarrow \Omega$$

$$\phi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A . \end{cases}$$

Note: In general, if $t : T \rightarrowtail \Omega$ is a subobject classifier,

$$t \rightsquigarrow \operatorname{Hom}(D, \Omega) \xrightarrow{\sim} \operatorname{Sub}(D) \qquad (*)$$

$$D \xrightarrow{\varphi} \Omega \longmapsto [\varphi^* t].$$

Prop $\forall$ small $\mathcal{C}$, $\mathbf{Set}^{\mathcal{C}^{op}}$ has a subobject classifier.

Pf If $t : T \rightarrowtail \Omega$ exists, one must have

$$\Omega(C) \cong \operatorname{Hom}(y(C), \Omega) \cong \operatorname{Sub}(y(C)) = \operatorname{Sieve}(C).$$

Now, let $T = 1$ and
$$t : 1 \longrightarrow \Omega$$
$$t(C) : 1 \longrightarrow \operatorname{Sieve}(C)$$
$$* \longmapsto \max(C).$$

Let us show that $t : T \rightarrowtail \Omega$ is a subobject classifier.

Let $Y \rightarrowtail X$ be a subpresheaf, and suppose $\phi_Y : X \longrightarrow \Omega$

classifies Y. $\Rightarrow$

$$\begin{tikzcd} Y(C) \arrow[r] \arrow[d, tail] & * \arrow[d, "\max(C)"] \\ X(C) \arrow[r, "(\phi_Y)(C)"'] & \operatorname{Sieve}(C) \end{tikzcd}$$

$\forall C$

$\Rightarrow x \in X(C) \Leftrightarrow (\phi_Y)_C(x) = \max(C) \Leftrightarrow id_C \in \phi_{Y_C}(x)$

By naturality: $\forall\ f: c' \to c$

$$
\begin{tikzcd}
x \in X(c) \arrow[r, "(\phi_Y)_c"] \arrow[d, "X(f)"'] & \Omega(c) \arrow[d, "f^*"] \\
X(c') \arrow[r, "(\phi_Y)_{c'}"'] & \Omega(c')
\end{tikzcd}
$$

$\Rightarrow X(f)(x) \in Y(c') \Leftrightarrow id_{c'} \in \phi^{Y}_{c'}(X(f)(x)) = f^* \phi^{Y}_{c}(x)$

$\Leftrightarrow f \in (\phi_Y)(c)(x)$

$\Rightarrow (\phi_Y)(c)(x) = \{ f: c' \to c \mid X(f)(x) \in Y(c') \}$ (*)

$\therefore\ \exists!\ \phi_Y : X \longrightarrow \Omega$ given by (*) s.t.

$$
\begin{tikzcd}
Y \arrow[r, "!"'] \arrow[d, tail] & 1 \arrow[d, "t"] \\
X \arrow[r, "\phi_Y"'] & \Omega
\end{tikzcd}
$$

If $\mathcal{E} = Sh_J(\mathcal{C}) \underset{i}{\overset{a}{\rightleftarrows}} Set^{\mathcal{C}^{op}}$, let

$$\Omega_J = a(\Omega).$$

HW: Ω_J is a subobject classifier for $\mathcal{E}$.

<u>Def</u> A category $\mathcal{E}$ is an <u>elementary</u> <u>topos</u> if

i) $\mathcal{E}$ has finite limits

ii) $\mathcal{E}$ is Cartesian closed

iii) $\mathcal{E}$ has a subobject classifier