

Categorical Properties of Topoi

1.

Let $\mathcal{E} = \text{Sh}_{\mathcal{J}}(\mathcal{C}) \xrightleftharpoons[i]{a} \text{Set}^{\mathcal{C}^{\text{op}}}$ be a Grothendieck topos.

$$a = (\cdot)^{++} \dashv i.$$

Prop: \mathcal{E} is both complete and cocomplete.

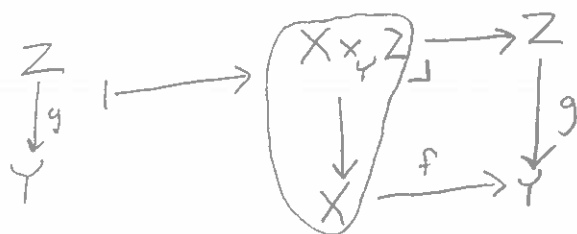
- Limits are computed in $\text{Set}^{\mathcal{C}^{\text{op}}}$

- To compute colimits, first compute them in $\text{Set}^{\mathcal{C}^{\text{op}}}$ and then apply a .

(This is a general fact about reflective subcategories).

Let \mathcal{D} be a category with pullbacks, and let $f: X \rightarrow Y \in \mathcal{D}$.

$$\leadsto f^*: \mathcal{D}/Y \longrightarrow \mathcal{D}/X$$



Def \mathcal{D} as above is a category in which colimits are universal

if for each $f: X \rightarrow Y$, f^* preserves small colimits.

This basically says that if $F: I \rightarrow \mathcal{D}$ is a small diagram,

and $\begin{matrix} \text{colim}_D F(i) \\ \downarrow g \end{matrix}$ is a diagram, then



$$X \times_Y (\text{colim}_D F(i)) \cong \text{colim}_D (X \times_Y F(i)).$$

Prop Colimits are universal in any (Groth.) topos \mathcal{E} .

2.

PF The fact that colimits are universal in \mathbf{Set} is an easy exercise. Since limits and colimits in $\mathbf{Set}^{\text{cop}}$ are computed "object-wise" it follows that colimits are also universal in $\mathbf{Set}^{\text{cop}}$.

Now suppose

$$\begin{array}{ccc}
 & \text{colim}_D F(D) & \\
 & \downarrow g & \\
 X & \xrightarrow{f} & Y
 \end{array}$$

is a diagram in $\mathcal{E} = \mathbf{Sh}_J(\mathcal{C})$ with $F: I \rightarrow \mathcal{E}$.

Consider

$$\begin{array}{ccc}
 & \text{colim}_D i(F(D)) & \\
 & \downarrow & \\
 iX & \longrightarrow & iY
 \end{array}$$

in $\mathbf{Set}^{\text{cop}}$, with $i: \mathcal{E} \rightarrow \mathbf{Set}^{\text{cop}}$, $a \circ i$.

Then $iX \times_{iY} (\text{colim}_D i(F(D))) \cong \text{colim}_D iX \times_{iY} i(F(D))$

apply $a \Rightarrow X \times_Y (\text{colim}_D F(D)) \cong \text{colim}_D X \times_Y F(D)$ since a preserve

colimits and pullbacks. \square

Def Let $\coprod_i A_i$ be a coproduct in a category \mathcal{C} with

$(S_j: A_j \rightarrow \coprod_i A_i)_j$ the associated cocone. This coproduct is said to be disjoint if

1) each S_j is a monomorphism

2) $\forall j, j'$ the pullback

$$\begin{array}{ccc}
 A_j \times_{\coprod_i A_i} A_{j'} & \longrightarrow & A_{j'} \\
 \downarrow & \lrcorner & \downarrow S_{j'} \\
 A_j & \xrightarrow{S_j} & \coprod_i A_i
 \end{array}$$

has $A_j \times_{\coprod_i A_i} A_{j'} = \emptyset$.

Prop In a (Groth) topos, all coproducts are disjoint.

3.

Pf. This is true for Set ($\coprod =$ disjoint union), and

\therefore for Set^{cop} . Since $a = (\cdot)^{++}$ preserves colimits and pullbacks, it follows that 2) holds. For 1) notice that

$S: B \rightarrow A$ is mono

\Updownarrow

$$\begin{array}{ccc} B & \xrightarrow{\text{id}} & B \\ \text{id} \downarrow & & \downarrow S \\ B & \xrightarrow{S} & A \end{array}$$

is a p.b diagram.

So 1) follows since a is left exact.

Def Let \mathcal{D} be a category and $D \in \mathcal{D}_0$. An equivalence relation is a subobject $R \rightrightarrows D \times D$ s.t. $\forall E \in \mathcal{D}_0$ the subset

$$\text{Hom}_{\mathcal{D}}(E, R) \subset \text{Hom}_{\mathcal{D}}(E, D) \times \text{Hom}_{\mathcal{D}}(E, D)$$

is an equivalence relation on the set $\text{Hom}_{\mathcal{D}}(E, D)$.

Given such an eq'l relation $R \rightrightarrows D \times D$, one may consider the induced pair of morphisms $R \rightrightarrows D$.

Def Given an eq'l relation $R \rightrightarrows D \times D$ as above, the quotient object (if it exists) is $D/R := \text{coeq}(R \rightrightarrows D)$.

Example: Let $f: X \rightarrow Y \in \mathcal{D}$ and suppose that \mathcal{D} has pullback

Consider $X \times_Y X \rightrightarrows X \times X$. This determines an eq'l relation on X .

Kernel pair = $X \times_Y X \rightrightarrows X$

Given an eq'l relation $R \rightrightarrows D \times D$ with a quotient object $D/R = \text{colim}(R \rightrightarrows D)$ consider the canonical map

$$p: D \longrightarrow D/R$$

and its kernel pair $D \times_{D/R} D \rightrightarrows D \times D$.

Def An equivalence relation $R \rightrightarrows D \times D$ is effective

$$\text{if } R = D \times_{D/R} D.$$

Note: In Set, every eq'l relation is effective.

Prop In a (Groth.) topos \mathcal{E} , every eq'l relation is effective.

Pf Let $\mathcal{E} \xleftarrow{i} \text{Set}^{\text{cop}}$, and $R \rightrightarrows F \times F \in \mathcal{E}$ an eq'l rel.

Then $iR \rightrightarrows iF \times iF$ is an eq'l rel in Set^{cop} . Since effectivity involves taking colimits and limits, which can be computed object-wise in Set, it follows that all eq'l relations in Set^{cop} are effective.

Suppose $q: iF \longrightarrow Q = \text{coeq}(iR \rightrightarrows iF)$. Then since eq'l relations in Set^{cop} are effective,

$$\begin{array}{ccc} iF \times_Q iF & \longrightarrow & iF \\ \downarrow & & \downarrow q \\ iF & \xrightarrow{q} & Q \end{array} \quad , \quad \begin{array}{ccc} iR: iF \times_Q iF & \rightrightarrows & iF \times iF \\ \downarrow \tau & \searrow \alpha & \downarrow \\ iR & \rightrightarrows & \star \end{array}$$

Applying $a: \text{coeq} \rightarrow \text{coeq}$: $aq: F \longrightarrow aQ = \text{coeq}(R \rightrightarrows F) = F/R$, and

$$\star \Rightarrow \begin{array}{ccc} F \times_{aQ} F & \xrightarrow{aq} & R \\ & \searrow & \downarrow \\ & & F \times F \end{array} \Rightarrow R \text{ is effective.}$$

Def A category \mathcal{D} with small colimits and pullbacks is locally κ -presentable, for a regular cardinal κ if, \exists a small subcategory $\mathcal{D}' \xrightarrow{\ell} \mathcal{D}$ s.t.

- i) $\text{Lan}_{\ell} = \text{id}_{\mathcal{D}}$ (\mathcal{D}' strongly generates \mathcal{D})
- ii) each $G \in \mathcal{D}'_0$ is κ -compact: \hookrightarrow means $\forall D_i, D = \text{colim}_{D' \in \mathcal{D}'} D_i, D' \rightarrow D, D' \in \mathcal{D}'$

$\text{Hom}(G, \cdot) : \mathcal{D} \rightarrow \text{Set}$
 preserves κ -filtered colimits. It is called just locally pres. if $\exists \kappa$ s.t. ..

Example Let $\mathcal{D} = \text{Set}^{\mathcal{C}^{op}}$. Then \mathcal{D} is locally κ -presentable

$\forall \kappa : \mathcal{D}' = \mathcal{C} \xrightarrow{y} \text{Set}^{\mathcal{C}^{op}}$

$\forall C \in \mathcal{C}, \text{Hom}(y(C), \cdot) : \text{Set}^{\mathcal{C}^{op}} \rightarrow \text{Set}$
 $\text{ev}_C : F \mapsto \text{Hom}(y(C), F) \cong F(C)$

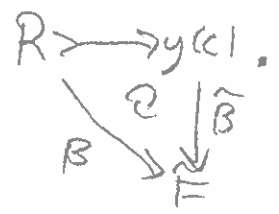
preserves all colimits.

Prop $\mathcal{E} = \text{Sh}_{\mathcal{I}}(\mathcal{C}) \xleftarrow{i} \text{Set}^{\mathcal{C}^{op}}$ (with \mathcal{C} small) is locally α -presentable, $\forall \alpha > |\mathcal{C}|$.

(1) i preserves α -filtered colimits:

Suffices to show that if $F = \text{colim}_{\mathcal{I}} F_k$ is α -filtered, then the colimit can be computed object-wise. Let $\tilde{F} = \text{colim}_{\mathcal{I}} i(F_k)$.

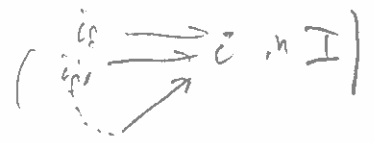
It suffices to show \tilde{F} is a sheaf. Suppose $R \rightrightarrows y(\mathcal{C})$ is a covering sieve and $\beta : R \rightarrow \tilde{F}$, wts $\exists ! \tilde{\beta} : y(\mathcal{C}) \rightarrow \tilde{F}$ s.t.



$$\forall D \in \mathcal{C}_\alpha, \beta_D: R(D) \longrightarrow F(D) = \underset{I}{\text{colim}} F_K(D) = \prod_{K \in I} F_K(D) \quad (6)$$

$$f \longmapsto \beta_D(f) = [a_f]$$

for $a_f \in F_{i_f}(D)$ for some $i_f \in I$.



But since I is α -filtered and $\alpha > |\mathcal{C}_\alpha| \Rightarrow \exists i \in I$.

and $\tilde{a}_f \in F_i(D) \forall f \in R(D)$ s.t. $[a_f] = \beta_D(f) \forall f$.

Now consider $g: E \rightarrow D$, and

$$\begin{array}{ccc} R(D) & \xrightarrow{\beta_D} & F(D) \\ R(g) \downarrow & \circlearrowleft & \downarrow F(g) \\ R(E) & \xrightarrow{\beta_E} & F(E) \end{array}$$

$F_i(g)(\tilde{a}_f) \in F_i(E)$
 $\tilde{a}_{f \circ g} \in F_i(E)$

(*)

$$\Rightarrow F(g)([a_f]) = F(g)(\beta_D(f)) = \beta_E(R(g)(f)) = [a_{f \circ g}] \text{ holds in } F(E)$$

\Rightarrow it holds in $F_{j_f}(E)$ for some $j_f \in I$, and $\alpha > |\mathcal{C}_\alpha| \Rightarrow$

$\exists j$ s.t. (*) holds in $F_j \forall f$.

$$\text{Let } \gamma_D(f) := [a_f] \in F_j(D) \rightsquigarrow \gamma: R \rightarrow iF_j$$

s.t. $R \xrightarrow{\gamma} iF_j$. But F_j is a sheaf so $\exists!$

$$\begin{array}{ccc} R & \xrightarrow{\gamma} & iF_j \\ \beta \searrow & \circlearrowleft & \downarrow \\ & & F \end{array}$$

$$\begin{array}{ccc} R & \xrightarrow{\gamma} & iF_j \\ \downarrow & \circlearrowleft & \nearrow \tilde{\gamma} \\ y(C) & & \end{array}$$

Let $\tilde{\beta} = y(C) \xrightarrow{\tilde{\gamma}} iF_j \rightarrow F$. Uniqueness can be shown in a similar way.

(2) $\forall C$, $y(C)$ is α -compact in \mathcal{E} .

Let $F = \underset{I}{\text{colim}} F_K$ be α -filtered.

Then

$$\begin{aligned}
 \text{Hom}(aycc), \underbrace{\text{colim}}_I F_k &\cong \text{Hom}(ycc), i(\underbrace{\text{colim}}_I F_k) \\
 &\stackrel{\textcircled{1}}{\cong} \text{Hom}(ycc), \underbrace{\text{colim}}_I i(F_k) \\
 &\cong (\underbrace{\text{colim}}_I i(F_k)(c)) \\
 &\cong \underbrace{\text{colim}}_I \text{Hom}(ycc, i(F_k)) \\
 &\stackrel{+}{\cong} \underbrace{\text{colim}}_I \text{Hom}(aycc, F_k)
 \end{aligned}$$

$$\textcircled{3} \quad \forall E \in \mathcal{E}, \quad iE = \underbrace{\text{colim}}_{ycc \rightarrow iE} ycc \xrightarrow{a_{iE}} E \cong \underbrace{\text{colim}}_{ycc \rightarrow iE} aycc \cong \underbrace{\text{colim}}_{aycc \rightarrow E} aycc$$

So, if $\mathcal{E}' = \{aycc, cc \in \mathcal{E}\} \xrightarrow{e} \mathcal{E}$,
 $\text{Lan}_e \mathcal{E} \cong \text{id}_{\mathcal{E}}$, and each object of \mathcal{E}' is α -compact.

Theorem (Giraud's thm)

A category \mathcal{E} is a Grothendieck topos if and only if:

- i) \mathcal{E} is locally presentable
- ii) colimits in \mathcal{E} are universal
- iii) coproducts in \mathcal{E} are disjoint
- iv) every equivalence relation in \mathcal{E} is effective.

If \mathcal{C} is not small (or not \mathcal{U} -small) then
 i) may fail but ii) - iv) still hold.

Useful Criterion:

Suppose $\mathcal{E} \cong \text{Sh}_{\mathcal{J}}(\mathcal{C})$, \mathcal{J} coming from a Groth. pretopology,
 with \mathcal{C} not necc. \mathcal{U} -small.

Def $(\mathcal{C}, \mathcal{J})$ is a \mathcal{U} -site if \exists a \mathcal{U} -small set

$\mathcal{G} = \{G_\alpha \in \mathcal{C}_0\}$ s.t. $\forall C \in \mathcal{C} \exists$ a cover
 $(f_{\alpha_j} : G_{\alpha_j} \rightarrow C)$ of C . (G_α are called topological generators).

Thm If $(\mathcal{C}, \mathcal{J})$ is a \mathcal{U} -site, $\text{Sh}_{\mathcal{J}}(\mathcal{C})$ is a topos
 (in \mathcal{U}).

E.g. $\mathcal{C} = \text{Mfd}$, $\mathcal{G} = \{\mathbb{R}^n \mid n \geq 0\}$.

Other Properties of Topoi

Cartesian closure:

Def A category \mathcal{S} is Cartesian-closed if it has binary products, and for each object $D \in \mathcal{S}_0$, the functor

$$(-) \times D : \mathcal{S} \longrightarrow \mathcal{S}$$

$$E \longmapsto E \times D$$

has a right adjoint $(-)^D : \mathcal{S} \longrightarrow \mathcal{S}$

$$E \longmapsto E^D$$

Example 1 $\mathcal{S} = \text{Set}$, $E^D := \text{Hom}(D, E)$:

\forall sets A , $\text{Hom}(A \times D, E) \xrightarrow{\sim} \text{Hom}(A, E^D)$

$$H : A \times D \rightarrow E \sim \tilde{H} : A \rightarrow E^D$$

$$\tilde{H}(a) : D \rightarrow E$$

$$\tilde{H}(a)(d) = H(a, d).$$

Example 2 $\mathcal{S} = \text{Set}^{e^{\text{op}}} \ni F, G$.

\swarrow Yoneda

$$FG(C) \cong \text{Hom}(y(C), FG) \cong \text{Hom}(y(C) \times G, F)$$

this must define FG if it exists. Now, suppose $H \in \text{Set}^{e^{\text{op}}}$

$$\text{Hom}(H, FG) = \text{Hom}(\text{colim}_{y(C) \rightarrow H} y(C), FG)$$

$$\cong \lim_{\longleftarrow y(C) \rightarrow H} \text{Hom}(y(C) \times G, F)$$

$$\cong \text{Hom}(\text{colim}_{y(C) \rightarrow H} (y(C) \times G), F)$$

\swarrow colimits universal

$$\cong \text{Hom}((\text{colim}_{y(C) \rightarrow H} y(C)) \times G, F)$$

$$\cong \text{Hom}(H \times G, F).$$

Prop If $F \in \text{Sh}_J(\mathcal{C})$ and $G \in \text{Set}^{\mathcal{C}^{\text{op}}}$, then

$(iF)^G \in \text{Sh}_J(\mathcal{C})$.

PF Let $R \rightarrow y(c) \in \text{Cov}_J(\mathcal{C})$. Then $\text{ar}: aR \xrightarrow{\sim} y(y(c))$.

Suppose
$$\begin{array}{ccc} f: R \rightarrow (iF)^G \\ \tilde{f}: R \times G \rightarrow iF \end{array} \xrightarrow{a} \begin{array}{ccc} a(R \times G) \xrightarrow{a\tilde{f}} a(iF) \cong F \\ \cong \downarrow \cong \\ aR \times aG \cong a(y(c)) \times aG \\ \cong \downarrow \cong \\ a(y(c)) \times G \end{array}$$

This
$$\begin{array}{ccc} a(y(c) \times G) \rightarrow F \\ y(c) \times G \rightarrow iF \\ \hline y(c) \rightarrow (iF)^G \\ \tilde{f} \uparrow \end{array}$$

s.t.
$$\begin{array}{ccc} R \xrightarrow{r} y(c) & \circ \\ \downarrow f & \downarrow \tilde{f} \\ & (iF)^G \end{array}$$

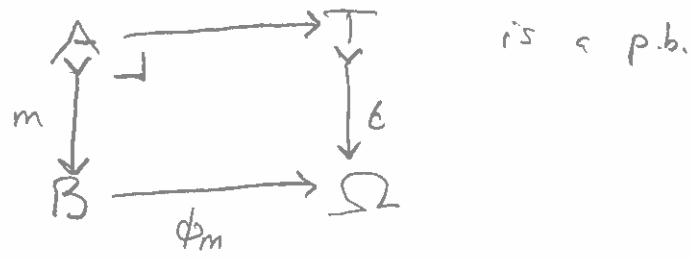
Easy to check that \tilde{f} must be curique.

Cor Every Grothendieck topos is Cartesian closed.

Subobject Classifiers

Def A subobject classifier in a category \mathcal{C} is a monomorphism $\epsilon: T \rightarrow \Omega$ s.t. \forall monomorphisms

$m: A \rightarrow B \exists! \phi_m: B \rightarrow \Omega$ s.t.



(so ϵ is the "universal subobject")

Example In Set, let $\Omega = \{0, 1\}$ 2-element set

and $\epsilon: 1 \rightarrow \Omega$
 $*1 \rightarrow 1$

Given $A \subset B$ $\exists!$ characteristic function

$$\phi_A: B \rightarrow \Omega$$
$$\phi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Note: In general, if $\epsilon: T \rightarrow \Omega$ is a subobject classifier,

$$\epsilon \rightsquigarrow \text{Hom}(D, \Omega) \xrightarrow{\sim} \text{Sub}(D) \quad (C \neq 1)$$
$$D \xrightarrow{\varphi} \Omega \longmapsto [\varphi * \epsilon]$$

Prop \forall small C , $\text{Set}^{C^{\text{op}}}$ has a subobject classifier.

Pf If $\epsilon: T \rightarrow \Omega$ exists, one must have

$$\Omega(C) \cong \text{Hom}(y(C), \Omega) \cong \text{Sub}(y(C)) = \text{Sieve}(C).$$

Now, let $T = 1$ and $\epsilon: 1 \rightarrow \Omega$
 $\epsilon(C): 1 \rightarrow \text{Sieve}(C)$
 $*1 \rightarrow \text{max}(C).$

Let us show that $\epsilon: T \rightarrow \Omega$ is a subobject classifier.

Let $Y \rightarrow X$ be a subprecheat, and suppose $\phi_Y: X \rightarrow \Omega$

classifies Y . \Rightarrow

$$\begin{array}{ccc} Y(C) & \xrightarrow{\quad} & * \\ \downarrow & \lrcorner & \downarrow \text{max}(C) \\ X(C) & \xrightarrow{(\phi_Y)_C} & \text{Sieve}(C) \end{array}$$

$\forall C$
 $\Rightarrow x \in X(C) \Leftrightarrow (\phi_Y)_C(x) = \text{max}(C) \Leftrightarrow \text{id}_C \in \phi_Y(x)$

By naturality: $\forall f: C' \rightarrow C$

$$\begin{array}{ccc} x \in X(C) & \xrightarrow{(\phi_Y)_C} & \Omega(C) \\ X(f) \downarrow & & \downarrow f^* \\ X(C') & \xrightarrow{(\phi_Y)_{C'}} & \Omega(C') \end{array}$$

$$\Rightarrow X(f)(x) \in Y(C') \Leftrightarrow \text{id}_{C'} \in \phi_{C'}^Y(X(f)(x)) = f^* \phi_{C'}^Y(x)$$

$$\Leftrightarrow f \in (\phi_Y)_C(C)(x)$$

$$\Rightarrow (\phi_Y)_C(C)(x) = \{ f: C' \rightarrow C \mid X(f)(x) \in Y(C') \} \quad (x)$$

$\therefore \exists! \phi_Y: X \rightarrow \Omega$ given by (x) s.t.

$$\begin{array}{ccc} Y & \xrightarrow{\eta} & 1 \\ \downarrow & & \downarrow \epsilon \\ X & \xrightarrow{\phi_Y} & \Omega \end{array}$$

If $\mathcal{E} = \text{Sh}_{\mathcal{J}}(\mathcal{C}) \xleftarrow{q} \text{Set}^{\mathcal{C}^{\text{op}}} \xrightarrow{i}$, let

$$\Omega_{\mathcal{J}} = a(\Omega)$$

HW: $\Omega_{\mathcal{J}}$ is a subobject classifier for \mathcal{E} .

Def A category \mathcal{E} is an elementary topos if

- i) \mathcal{E} has finite limits
- ii) \mathcal{E} is Cartesian closed
- iii) \mathcal{E} has a subobject classifier