**Def** A <u>basis for a Grothendieck topology</u> on a category $\mathcal{C}$ (sometimes called a Grothendieck pre-topology) is an assignment to each object $C \in \mathcal{C}_0$ a collection $\mathcal{B}(C)$ of families of arrows $(f_i : U_i \to C)_{i \in I}$, called <u>covering families</u> s.t.

1) If $g : D \xrightarrow{\sim} C$ is an isomorphism, then the singleton family $(g : D \xrightarrow{\sim} C) \in \mathcal{B}(C)$

2) If $(U_i \to C)_{i \in I} \in \mathcal{B}(C)$ and $f : D \to C$, then $\forall i$ each pull back

$$\begin{tikzcd} U_i \times_C D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "f"] \\ U_i \arrow[r] & C \end{tikzcd}$$

exists in $\mathcal{C}$, and $(U_i \times_C D \to D)_{i \in I} \in \mathcal{B}(D)$

3) If $(U_i \to C)_i \in \mathcal{B}(C)$, and $\forall i$ $(V_{ij} \to U_i)_j \in \mathcal{B}(U_i)$, then the collection

$$(V_{ij} \to C)_{i,j} \in \mathcal{B}(C).$$

A presheaf $F : \mathcal{C}^{op} \to \mathrm{Set}$ is a <u>sheaf</u> wrt $\mathcal{B}$ if $\forall$ $C \in \mathcal{C}_0$ & $\forall$ $(U_i \to C)_i \in \mathcal{B}(C)$, the induced map

$$(*) \quad F(C) \longrightarrow \underleftarrow{\lim} \left[ \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_C U_j) \right]$$

is an isomorphism. $F$ is <u>separated</u> if $(*)$ is mono.

Examples:

1) For any $\mathcal{C}$, let $B(C)$, $\forall C \in \mathcal{C}_0$ consist of all isomorphisms $D \xrightarrow{\sim} C$. Every presheaf is a sheaf.

2) For any $\mathcal{C}$, let $B(C)$ consist of all morphisms $D \to C$. The only sheaves are the terminal ones.

3) Let $\mathcal{C} = \mathcal{O}(X)$, $X$ a topological space, and let $B(U) = \{(U_\alpha \hookrightarrow U) \mid \bigcup_\alpha U_\alpha = U\}$. Sheaves are the usual ones. This the "open cover topology" on $X$

4) Let $\mathcal{C} = \mathrm{Top}$, $X$ a space,

$$B(X) = \{ ((X_\alpha \overset{f_\alpha}{\hookrightarrow} X)_\alpha \text{ open embeddings} \mid \bigcup_\alpha f_\alpha(X_\alpha) = X \}.$$

The sheaves are the usual ones. This is the (global) "open cover" topology.

5) Let $\mathcal{C} = \mathrm{Top}$, $X$ a space (or e.g. $\mathcal{C} = \mathrm{Mfd}$)

$$B(X) = \{ (X_i \xrightarrow{f_i} X) \mid \forall i \; f_i \text{ is a local homeo., and } \bigcup_i f_i(X_i) = X \}.$$

We will see that the sheaves are the same as 4).

6). Let $\mathcal{C} = \mathrm{Mfd}$ = cat. of smooth manifolds.

$$B(M) = \{ (f : N \to M) \text{ (one map)} \mid f \text{ is a surjective submersion} \}.$$

The sheaves are also the same as for open covers.

7) Let $\mathcal{C} = \mathrm{Set}$, and let $B(X) = \{(f : Y \twoheadrightarrow X)\}$ = singleton surjections.

The sheaves turn out to be only the representable presheaves.

Rmk A Groth. pre-topology is said to be <u>subcanonical</u> if every representable presheaf is a sheaf.

It is easy to check that this is eq'l to:

$\forall \; C \in \mathcal{C}_0$ and all $(f_i : U_i \to C)_i \in B(C)$,

$$\varinjlim\Big( \coprod_{i,j} U_i \times_C U_j \rightrightarrows \coprod_i U_i \Big) \cong C.$$

In most applications, one only works with subcanonical (pre)-topologies.

**Def** Given $D \in \mathcal{D}_0$, $\mathcal{D}$ a category, a subobject of $D$ is an equivalence class of a monomorphism $S \xrightarrow{f} D$, where $f \sim f' \iff$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\exists s"'] \arrow[dr, phantom, "\circlearrowleft"] & D \\ S' \arrow[ur, "f'"'] & \end{tikzcd}$$

We denote this by $[S] \rightarrowtail D$.

We will often abuse notation and identify subobjects with monomorphisms representing them.

Note Monos are stable under pullback, so if $S \hookrightarrow D$ is a mono, and $g : D' \longrightarrow D$,

$$\begin{tikzcd} D' \times_D S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d] \\ D' \arrow[r, "g"] & D \end{tikzcd}$$

mono

$\rightsquigarrow$ induces well defined functor $g^* : Sub(D) \longrightarrow Sub(D')$, where $Sub(D)$ is regarded as a poset by "inclusion" " subobjects.
(resp. $Sub(D')$)

**Def** Let $C \in \mathcal{C}_0$, $\mathcal{C}$ a category. A sieve on $C$ is a subobject $S \rightarrowtail y(C)$ in $Set^{\mathcal{C}^{op}}$.

Such subobjects can be defined concretely &

(really its image!)

Let $C' \in \mathcal{C}_0$, $S(C') \subset Hom(C', C) = y(C)(C')$

$$\begin{array}{ccc} C' & S(C') \subset Hom(C',C) & \exists f : C' \to C \\ \uparrow g & \downarrow S(g) \quad\quad \downarrow & \downarrow \\ C'' & S(C'') \subset Hom(C'', C) & (C'' \xrightarrow{g} C' \xrightarrow{f} C) \\ & & fg \in S(C''). \end{array}$$

So such a subobject is the data as a collection $\mathcal{S}$ of arrows in $\mathcal{C}$ with codomain $C$, s.t. if $(f : C' \to C) \in \mathcal{S}$ and $g : C'' \to C'$, $fg \in \mathcal{S}$.

## Examples

1) $C \in \mathcal{C}$, $\max(C) = [\mathrm{id}_{y(C)}]' \leftrightsquigarrow$ all arrows with codomain $C$.

2) Let $\mathcal{U} = (f_i : U_i \to C)_{i \in I}$ be any collection of maps.

Let $S_{\mathcal{U}}(C') = \left\{ g : C' \to C \;\middle|\; g \text{ factors as } C' \to U_i \xrightarrow{f_i} C \text{ for some } i \in I \right\}$

$\cap$

$y(C)(C')$

$S_{\mathcal{U}} \rightarrowtail y(C)$ is a well defined sieve.

3) $\min(C) = [\emptyset \to y(C)] \leftrightsquigarrow \mathcal{S} = \{\emptyset \to y(C)\}$

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**Def** A *Grothendieck topology* $J$ on a category $\mathcal{C}$, is an assignment to every object $C$ of $\mathcal{C}$ a set $\mathrm{Cov}_J(C)$ of sieves on $C$, called *covering sieves*, s.t.

1) $\max(C) \in \mathrm{Cov}_J(C)$

2) $R \in \mathrm{Cov}_J(C)$, $f : D \to C$, $\Rightarrow f^*R \in \mathrm{Cov}_J(D)$

3) If $R$ is a sieve on $C$, and $S$ is a covering sieve on $C$ s.t. $\forall D$, $\forall f \in S(D) \subseteq \mathrm{Hom}(D, C)$, $f^*R \in \mathrm{Cov}_J(D) \Rightarrow R \in \mathrm{Cov}_J(C)$.

A pair $(\mathcal{C}, J)$ of this form is called a Grothendieck site.

**Def** A sheaf on a Grothendieck site $(\mathcal{C}, J)$ is a presheaf $F: \mathcal{C}^{op} \longrightarrow Set$ s.t. $\forall$ $C$ and all $(S \rightarrowtail y(C)) \in Cov_J(C)$, the canonical map

$$(**) \quad F(C) \cong Hom(y(C), F) \longrightarrow Hom(S, F)$$

is an isomorphism.

Denote the full subcategory of $Set^{\mathcal{C}^{op}}$ on $J$-sheaves by $Sh_J(\mathcal{C})$.

Groth univ

**Def** A ((Grothendieck)) topos (in $\mathcal{U}$) is a category $\mathcal{E}$ equivalent to $Sh_J(\mathcal{C})$ for a ($\mathcal{U}$-) small Grothendieck site $(\mathcal{C}, J)$.

How does a basis $B$ on $\mathcal{C}$ generated a Groth. topology?

Define $Cov_{J(B)}(C) = \{ S \rightarrowtail y(C) \mid S_W \subset S$ for $W = (f_i : V_i \to C)_{i \in I} \in B(C) \}$

In the homework, you will show that $J(B)$ is a Groth. topology.

<u>Lemma</u>: Suppose $\mathcal{B}$ is a Groth. pre-topology on $\mathcal{C}$ and $F$ is separated. Suppose $S_{\mathcal{W}} \overset{j}{\rightarrowtail} S$, for $\mathcal{W} = \{f_i : U_i \to C\}$ a cover of $C$. Then the induced map

$$\circledast \qquad \mathrm{Hom}(S,F) \xrightarrow{j^*} \mathrm{Hom}(S_{\mathcal{W}}, F) \text{ is injective.}$$

<u>Pf</u>: Let $\varphi, \psi$ s.t. $j^*\varphi = j^*\psi$. Let $T \in \mathcal{C}_0$ and

$$(T \xrightarrow{h} C) \in S(T) \subset \mathrm{Hom}(T,C) \text{ be arbitrary.}$$

It suffices to show $\varphi_T(h) = \psi_T(h)$ (where $\varphi, \psi : S \Rightarrow F$).

Consider the pullback diagram

$$\begin{tikzcd}
T \times_C U_i \arrow[r, "h_i"] \arrow[d, "\lambda_i"'] \arrow[dr, "=: g_i"] \arrow[dr, phantom, very near start, "\lrcorner"] & U_i \arrow[d, "f_i"] \\
T \arrow[r, "h"'] & C
\end{tikzcd}$$

$(T \times_C U_i \xrightarrow{\lambda_i} T) \in \mathcal{B}(T)$

Also, $\forall\, i$,

$(T \times_C U_i \xrightarrow{g_i} C) \in S_{\mathcal{W}}(T \times_C U_i) \subset S(T \times_C U_i)$

since $g_i = f_i \circ h_i$.

$\therefore$ by $\circledast$, $\varphi_{T \times_C U_i}(g_i) = \psi_{T \times_C U_i}(g_i) \;\; \forall\, i.$ $\quad$ (***)

Note that the following diagram commutes by naturality:

$$\begin{tikzcd}
h \in S(T) \arrow[r, "\varphi_T"] \arrow[d, "S(\lambda_i)"'] \arrow[dr, phantom, "\circlearrowright"] & F(T) \arrow[d, "F(\lambda_i)"] \\
g_i \in S(T \times_C U_i) \arrow[r, "\varphi_{T \times_C U_i}"'] & F(T \times_C U_i)
\end{tikzcd}$$

and similarly for $\psi$.

So (***) $\Rightarrow$

$F(\lambda_i)\varphi_T(h) = F(\lambda_i)\psi_T(h)$

$\forall\, i$.

I.e.

$$F(T) \longrightarrow \lim\Big[\prod_i F(T \times_C U_i) \rightrightarrows \prod_{i,j} F(T \times_C U_i \times_C U_j)\Big]$$

$$\varphi_T(h) \longmapsto \prod_i F(\lambda_i)(\varphi_T(h))$$

$$\qquad\qquad\qquad\qquad \shortparallel$$

$$\psi_T(h) \longmapsto \prod_i F(\lambda_i)(\psi_T(h))$$

$\Rightarrow \varphi_T(h) = \psi_T(h)$ since $F$ is separated.

<u>Lemma</u>: Given $W = (f_i : U_i \to C)_{i\in I}$,

$$S_W = \varinjlim \Big[ \coprod_{i,j} y(U_i \times_C U_j) \rightrightarrows \coprod_i y(U_i) \Big].$$

<u>Pf.</u> Suffices to show $\forall\ D \in \mathcal{C}_0$ that

$$\text{☆} \quad S_W(D) = \varinjlim \Big[ \coprod_{i,j} \mathrm{Hom}(D, U_i \times_C U_j) \rightrightarrows \coprod_i \mathrm{Hom}(D, U_i) \Big].$$

Note that there is a canonical surjection:

$$\coprod_i \mathrm{Hom}(D, U_i) \twoheadrightarrow S_W(D) = \{\varphi : D \to C \mid \varphi = \gamma_i \circ f_i,\ \gamma_i : D \to U_i, \text{ for some}$$
$$\lambda_i \longmapsto f_i \circ \lambda_i.$$

$$\Rightarrow S_W(D) \cong \coprod_i \mathrm{Hom}(D, U_i) / \sim \quad \text{where } \lambda_i \sim \lambda_j \Leftrightarrow f_i \circ \lambda_i = f_j \circ \lambda_j.$$

$\Leftrightarrow$

$$\begin{tikzcd}
D \arrow[r, dashed, "{(\lambda_i, \lambda_j)}"] \arrow[rr, bend left, "\lambda_i"] \arrow[dr, bend right, "\lambda_j"'] & U_i \times_C U_j \arrow[r, "pr_j"] \arrow[d, "pr_i"] & U_j \arrow[d, "f_j"] \\
 & U_i \arrow[r, "f_i"] & C
\end{tikzcd}$$

$$\Leftrightarrow \exists(!)\ \theta : D \to U_i \times_C U_j \text{ s.t. } \begin{array}{l} pr_i \circ \theta = \lambda_i \\ pr_j \circ \theta = \lambda_j \end{array} \Rightarrow \text{☆}. \quad \square$$

$\Rightarrow S_W(D)$

Corollary: $F : \mathcal{C}^{op} \to Set$ is a sheaf for the pre-topology $B \iff F$ is a $J(B)$-sheaf.

Pf Suppose $F$ is a $J(B)$-sheaf and let $\mathcal{W} = (f_i : U_i \to C)_i \in B(C)$. Then $S_{\mathcal{W}} \in Cov_{J(B)}(C) \Rightarrow$

$$F(C) \cong Hom(y(C), F) \xrightarrow{\sim} Hom(S_{\mathcal{W}}, F)$$
$$= Hom(\underrightarrow{colim} [\coprod_{i,j} y(U_i \times_C U_j) \rightrightarrows \coprod_i y(U_i)], F)$$
$$\cong \underleftarrow{lim} [\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_C U_j)],$$

so $F$ is a sheaf for $B$.

Conversely, suppose that $F$ is a sheaf for $B$. Let $S \in Cov(C)$, so $\exists\ \mathcal{W} = (U_i \to C)_i \in B(C)$ with $S_{\mathcal{W}} \overset{j}{\subset} S \subset y(C)$.

Consider the composite

$$\begin{tikzcd} Hom(y(C), F) \arrow[r, "\alpha"] \arrow[rr, bend right, "(*)\ \sim"'] & Hom(S, F) \arrow[r, "j^*"] & Hom(S_{\mathcal{W}}, F) \cong \underleftarrow{lim} [\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_C U_j)] \end{tikzcd}$$

Since $(*)$ is surj. $\Rightarrow j^*$ is, but since $F$ is separated $\Rightarrow j^*$ is injective, hence an iso $\Rightarrow \alpha$ is an iso. $\square$

Rmk Similarly, one shows that $F$ is separated $\iff$ $\forall\ C \in \mathcal{C}_0$ and all $S \in Cov(C)$,

$$F(C) \cong Hom(y(C), F) \longrightarrow Hom(S, F) \text{ is injective.}$$