Sheaves on a Category of Spaces

Let $\mathcal{S} \subseteq Top$ be a full subcategory of topological spaces closed under taking open subspaces.

<u>Def</u> A presheaf $F : \mathcal{S}^{op} \longrightarrow Set$ is a <u>sheaf</u>

$\Leftrightarrow$ $\forall$ spaces $T \in \mathcal{S}_0$, and all open covers $\{U_\alpha \hookrightarrow T\}_{\alpha \in I}$,

the induced map

$$F(T) \longrightarrow \varprojlim \left[\prod_{\alpha \in I} F(U_\alpha) \rightrightarrows \prod_{(\alpha,\beta) \in I^2} F(U_\alpha \cap U_\beta) \right] \quad (*)$$

is an isomorphism. If $(*)$ is injective, F is is a <u>separated presheaf</u>. Denote the category of sheaves on $\mathcal{S}$ by $Sh(\mathcal{S})$.

$\forall$ $T \in \mathcal{S}_0$, $\exists$ a functor

$$\begin{aligned} p_T : \mathcal{O}(T) &\longrightarrow \mathcal{S} \\ U &\longmapsto U \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} p_T^* : Set^{\mathcal{S}^{op}} &\longrightarrow Set^{\mathcal{O}(T)^{op}} \\ F &\longmapsto F \circ p_T^{op} \end{aligned}$$

$F \in Set_0^{\mathcal{S}^{op}}$ is a sheaf $\Leftrightarrow$ $p_T^* F$ is a sheaf on T

$\forall$ spaces $T \in \mathcal{S}_0$.

<u>Note</u> The Yoneda embedding produces a full and faithful functor

$$y : \mathcal{S} \hookrightarrow Set^{\mathcal{S}^{op}}$$

If $T, X \in \mathcal{S}_0$ are spaces, $p_X^* y(T) = C(-, T) : \mathcal{O}(X)^{op} \to Set$ is a sheaf $\Rightarrow$ $y(T)$ is a sheaf $\forall\, T \in \mathcal{S}_0$.

Since y is full and faithful, we will abuse notation and write T for $y(T)$.

Using the Yoneda Lemma, we can rewrite $(*)$ as:

the induced map

$$(**) \quad Hom(T, F) \longrightarrow \varprojlim \left[\prod_{\alpha \in I} Hom(U_\alpha, F) \rightrightarrows \prod_{(\alpha,\beta) \in I^2} Hom(U_\alpha \cap U_\beta, F) \right]$$

$(**)$ is iso $\iff$ $\forall$ collections of maps $f_\alpha : U_\alpha \to F$ s.t. $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ $\forall\, \alpha, \beta$

$$\exists!\ f : T \to F \text{ s.t. } f|_{U_\alpha} = f_\alpha \ \forall\, \alpha.$$

In this sense, we can view sheaves $F \in Sh(\mathcal{S})$ as geometric objects that we can map into in a continuous way.

Observe also that the RHS of $(**)$ can be written as

$$Hom\left(\varinjlim{colim} \left(\coprod_{(\alpha,\beta) \in I^2} y(U_\alpha \cap U_\beta) \rightrightarrows \coprod_{\alpha \in I} y(U_\alpha) \right), F \right)$$

Notice that

$$T \cong \varinjlim\mathrm{colim}\Big(\coprod_{(\alpha,\beta)\in I^2} U_\alpha \cap U_\beta \rightrightarrows \coprod_{\alpha\in I} U_\alpha\Big). \quad (***)$$

The Yoneda embedding $y: \mathcal{S} \hookrightarrow Set^{\mathcal{S}^{op}}$ does not preserve colimits, (since $Set^{\mathcal{S}^{op}}$ is the free colimit cocompletion), but we do have a canonically induced map

$$\underrightarrow{\mathrm{colim}}\Big(\coprod_{(\alpha,\beta)\in I^2} y(U_\alpha\cap U_\beta) \rightrightarrows \coprod_{\alpha\in I} y(U_\alpha)\Big) =: S_{\mathcal{U}}$$

$$\Big\downarrow \text{✡}$$

$$y\Big(\underrightarrow{\mathrm{colim}}\Big(\coprod_{(\alpha,\beta)\in I^2} U_\alpha\cap U_\beta \rightrightarrows \coprod_{\alpha\in I} U_\alpha\Big)\Big) = y(T).$$

where $\mathcal{U} = \{(U_\alpha \hookrightarrow T)\}_{\alpha\in I}$. $F \in Set^{\mathcal{S}^{op}}$ is a sheaf $\Leftrightarrow$

$\forall$ maps of the form ✡, the induced morphism

$$\mathrm{Hom}(y(T), F) \xrightarrow{\text{✡}^*} \mathrm{Hom}(S_{\mathcal{U}}, F) \quad \text{is an isomorphism,}$$

which is to say "F views maps of the form ✡ as isos",

that is, F is a ✡-local object.

Another way of defining $Sh(\mathcal{S})$ is that it is the largest subcategory $\mathcal{D}$ of $Set^{\mathcal{S}^{op}}$ containing $y: \mathcal{S} \hookrightarrow Set^{\mathcal{S}^{op}}$ s.t. $y|_{\mathcal{D}}: \mathcal{S} \hookrightarrow \mathcal{D}$ preserves colimits of the form $(***)$.

<u>Def</u> Let T be a space. A presheaf

$$F : (Top/T)^{op} \longrightarrow Set \quad \text{is a } \underline{\text{sheaf}}$$

if $\forall$ $P \xrightarrow{\pi} T \in (Top/T)_0$ and all open covers $\{(U_\alpha \overset{i_\alpha}{\hookrightarrow} P)\}_{\alpha \in I}$,

the induced map

$$F(\pi) \longrightarrow \varprojlim \left[\prod_\alpha F(\pi \circ i_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(\pi \circ i_{\alpha,\beta}) \right]$$

is an iso.

<u>Note</u> F is a sheaf $\iff$ under the eq'l

$$Set^{Top^{op}}/T \simeq Set^{(Top/T)^{op}}$$

F corresponds to a map $\gamma : G \longrightarrow T$ in $Set^{Top^{op}}$ s.t.

G is a sheaf.

Warning: Need to be careful with size issues to make this precise.

So we can freely play with sheaves on large categories, we will

turn to Grothendieck:

skip for now?

Grothendieck Universes

<u>Def</u> A <u>Grothendieck universe</u> is a non-empty set $\mathcal{U}$ s.t.

1. If $X \in \mathcal{U}$, and $y \in X$ then $y \in \mathcal{U}$

(equivalently $X \in \mathcal{U} \Rightarrow X \subseteq \mathcal{U}$)

2. If $x, y \in \mathcal{U} \Rightarrow \{x, y\} \in \mathcal{U}$
3. If $X \in \mathcal{U} \Rightarrow P(X) \in \mathcal{U}$ (power set)
4. If $(X_i \in \mathcal{U}, i \in I)$ is a collection with $I \in \mathcal{U} \Rightarrow \bigcup_{i \in I} X_i \in \mathcal{U}$.

From these axioms it follows that:

- If $x \in \mathcal{U} \Rightarrow \{x\} \in \mathcal{U}$
- If $A \subseteq B \in \mathcal{U} \Rightarrow A \in \mathcal{U}$ (hence $\phi \in \mathcal{U}$, provided $\mathcal{U} \neq \phi$)
- If $x, y \in \mathcal{U} \Rightarrow \{x, \{x, y\}\} \in \mathcal{U}$
- If $x, y \in \mathcal{U} \Rightarrow x \cup y$ and $x \times y \in \mathcal{U}$.
- If $x, y \in \mathcal{U} \Rightarrow \mathrm{Hom}(x, y) \in \mathcal{U}$
- If $I \in \mathcal{U}$ and $(x_i \in \mathcal{U} \mid i \in I) \Rightarrow \prod_{i \in I} x_i \in \mathcal{U}$

  $\coprod_{i \in I} x_i \in \mathcal{U}$
- If $x \in \mathcal{U}$, $\bigcup_{t \in x} t \in \mathcal{U}$

Trivial examples: ϕ, $V_\omega = \bigcup_{k=0}^{\infty} V_k$, $V_k = P(V_{k-1})$, $V_0 = \phi$

hereditarily finite sets

We now add the following axiom to ZFC - Set theory:

<u>Universe axiom</u>: $\forall$ sets X $\exists$ a Grothendieck universe $\mathcal{U}$ s.t. $X \in \mathcal{U}$.

This axiom is equivalent to the <u>inaccessible cardinal axiom</u>:

$\forall$ cardinals κ $\exists$ a <u>strongly inaccessible cardinal</u> λ s.t. $\lambda > \kappa$.

A cardinal λ is said to be <u>strongly inaccessible</u> if

- λ is a <u>strong limit cardinal</u>, that is:

  $0 \neq \lambda$ and $\forall$ $\kappa < \lambda$, $2^\kappa < \lambda$

and

- λ is <u>regular</u>:

  if $\{X_\alpha\}_{\alpha \in A}$ with $|A| < \lambda$ and $|X_\alpha| < \lambda \ \forall \alpha$

  $\Rightarrow |\coprod_\alpha X_\alpha| < \lambda$. (e.g. ω)

We will fix a Grothendieck universe $\mathcal{U} \ni \mathbb{N}$, and a Grothendieck universe $\mathcal{V}$ s.t. $\mathcal{U} \in \mathcal{V}$ ($\Rightarrow \mathcal{U} \subset \mathcal{V}$).

Elements of $\mathcal{U}$ will be called $\mathcal{U}$-small.

$Set_{\mathcal{U}}$ = the full subcategory of all sets on $\mathcal{U}$-small sets. = Set (notation) "small sets"

$Set_{\mathcal{V}}$ = " " $\mathcal{V}$-small sets. = $\widehat{Set}$ "large sets"

<u>Def</u> A <u>$\mathcal{U}$-category</u> is a category $\mathcal{C}$ s.t. $\forall\, c, D \in \mathcal{C}$, $Hom(c, D)$ is (isomorphic to) an element of $\mathcal{U}$.

A category $\mathcal{C}$ is <u>$\mathcal{U}$-small</u> if $\mathcal{C}_0$ & $\mathcal{C}_1 \in \mathcal{U}$ (more formally, iso. to elements).

<u>Example</u> $Set_{\mathcal{U}}$ is a $\mathcal{U}$-category, but is not $\mathcal{U}$-small.

Basic facts:

$Set_{\mathcal{U}}$ has all $\mathcal{U}$-small limits and colimits.

(Behaves "just like" Set, but replace the phrase small with $\mathcal{U}$-small everywhere)

- Formally, $\mathcal{U}$ is a model of ZFC, so everything about Set provable in ZFC will be true in $Set_{\mathcal{U}}$.

- If $\mathcal{C}$ is $\mathcal{U}$-small, if $\mathcal{D}$ is $\mathcal{U}$-small

$\Rightarrow \mathcal{D}^{\mathcal{C}}$ is $\mathcal{U}$-small

if $\mathcal{D}$ is a $\mathcal{U}$-category

$\Rightarrow \mathcal{D}^{\mathcal{C}}$ is a $\mathcal{U}$-category.

How does this help us?

Replace Top with $Top_{\mathcal{U}} = \{X$ a top'l space s.t. $\underline{X}$ is $\mathcal{U}$-small$\}$.

Then, since $\mathcal{U} \in \mathcal{V} \Rightarrow Top_{\mathcal{U}}$ is $\mathcal{V}$-small.

<u>Notation</u>: $Top_{\mathcal{U}} = Top$, $Top_{\mathcal{V}} = \widehat{Top}$

$Set_{\mathcal{U}} = Set$, $Set_{\mathcal{V}} = \widehat{Set}$

Similarly for $Sh_{\mathcal{U}}(Top_{\mathcal{U}}) = Sh(Top)$

$Sh_{\mathcal{V}}(Top_{\mathcal{U}}) = \widehat{Sh}(Top)$

<u>Note</u> Top (really $Top_{\mathcal{U}}$) is actually small now, so we do have

e.g. $$Set^{Top^{op}}/T \simeq Set^{(Top/T)^{op}}$$