Stalks (Lecture 3)

<u>Def</u> Let $F : \mathcal{O}(X)^{op} \longrightarrow Set$ be any presheaf on X, and let $x \in X$. Let $x \in U \cap V$ - open. We say that $f \in F(U)$ and $g \in F(V)$ <u>have the same germ at x</u> if $\exists$ an open $x \in W \subset U \cap V$ s.t.

$$F(W \subset U)(f) = F(W \subset V)(g) \in F(W)$$
$$\text{" } f_{|W} \text{ "} \qquad \text{" } g_{|W} \text{ "}$$

This induces an eq'l relation on $\coprod_{\substack{U \ni x \\ \text{open}}} F(U)$.

Let $F_x := \coprod_{\substack{U \ni x \\ \text{open}}} F(U) / \sim$ (called the stalk of F @ x.) Denote the image of $f \in F(U)$ in F_x by $\text{germ}_x f$ - the germ of f at x.

<u>Ex</u> If $F = C(\circ, \mathbb{R})$, $F_x = C_x(X, \mathbb{R})$ = germs of cont. functions at x. (Hence the name). (Another example, $X = \mathbb{C}$, $F = \mathcal{O}_X$, $F_x \cong$ ring of convergent power series in $z - x$.)

Let $\mathcal{O}_x(X)$ be the full subcategory of $\mathcal{O}(X)$ on the open neighborhoods of x, and let $i_x : \mathcal{O}_x(X) \hookrightarrow \mathcal{O}(X)$ be the inclusion.

<u>Claim:</u> Taking germs induces a cocone

$$F \circ i_x \xRightarrow{\text{germ}_x} \Delta_{F_x} \quad (\text{happens in } Set^{\mathcal{O}_x^{op}})$$

<u>In words</u> "f and $f_{|U}$ have the same germ."

<u>Proposition</u> $germ_x(\cdot)$ is colimiting.

<u>Pf</u> WTS $\forall$ sets A, the canonical map

$$\mathrm{Hom}(F_x, A) \xrightarrow{\widehat{germ}_x} \mathrm{Cocone}(F\circ i_x, A) = \mathrm{Hom}(F\circ i_x, \Delta_A)$$
$$F_x \xrightarrow{f} A \longmapsto F\circ i_x \overset{germ_x}{\Longrightarrow} \Delta_{F_x} \overset{\Delta_f}{\Longrightarrow} \Delta_A$$

is an iso.

Let $\lambda : F\circ i_x \Longrightarrow \Delta_A$ be any cocone. Suppose $\exists\ \tilde{\lambda} : F_x \longrightarrow A$ s.t. $\widehat{germ}_x(\tilde{\lambda}) = \lambda$. Let $germ_x f \in F_x$, with $f \in F(U)$. Then

$$\begin{tikzcd} f \in F(U) \arrow[r, "germ_{x,U}"] \arrow[rr, bend right, "\lambda_U"'] & F_x \arrow[r, "\tilde{\lambda}"] \arrow[d, phantom, "\circlearrowleft"] & A \\ & {} & \end{tikzcd}$$

$\Rightarrow \tilde{\lambda}(germ_x f) = \lambda_U(f)$. $(*)$

So, for an arbitrary λ, if $\tilde{\lambda}$ exists, it is uniquely defined by $(*)$. To show $\tilde{\lambda}$ exists, just need to show $\tilde{\lambda}$ does not depend on the choice of f.

Suppose $g \in F(V)$ with $germ_x g = germ_x f \Rightarrow \exists\ W \subset U \cap V$ s.t. $W \ni x$, $g|_W = f|_W$. Since λ is natural!

$$\begin{tikzcd} f \in F(U) \arrow[dr] \arrow[drr, bend left, "\lambda_U"] & & \\ & F(W) \arrow[r, "\lambda_W"] & A \\ g \in F(V) \arrow[ur] \arrow[urr, bend right, "\lambda_V"'] & & \end{tikzcd}$$

$$\Rightarrow \lambda_U(f) = \lambda_W(f|_W)$$
$$\|$$
$$\lambda_V(g) = \lambda_W(g|_W)$$

$\square$

So $\boxed{F_x = \varinjlim_{x \in U} F(U).}$

<u>Ex</u> $U \in \mathcal{O}(X)$, $y(U)_x = \coprod_{x \in V \text{ open}} y(U)(V) / \sim$,

<u>Note</u> $\coprod_{x \in V} \overset{\mathrm{Hom}(V,U)}{y(U)(V)} \cong \coprod_{x \in V} \begin{cases} * & \text{if } V \subset U \\ \phi & \text{otherwise} \end{cases} \cong \coprod_{x \in V \subset U} \overset{\text{terminal}}{*} = \star$

if $x \notin U$ this is an empty coproduct, so $\phi \Rightarrow y(U)_x \cong \phi$.

If $x \in U$ $\star \cong \mathcal{O}_x(U) \setminus \phi$ = open subsets of U containing x $\neq \phi$

Let $*_V \longmapsto V$

Let $V, V' \in \mathcal{O}(U) \setminus \{\phi\}$, $\Rightarrow V \cap V' =: W \in \mathcal{O}(U)_x$ since $x \in V \cap V'$.

and

$$\begin{tikzcd}
*_V \in y(U)(V) \cong 1 \arrow[dr] \arrow[ddr, mapsto] & & *_{V'} \in y(U)(V') \cong 1 \arrow[dl] \arrow[ddl, mapsto] \\
& y(U)(W) \cong 1 & \\
& *_W &
\end{tikzcd}$$

$\Rightarrow V \sim V'$

$\Rightarrow y(U)_x \cong *$.

So $y(U)_x \cong \begin{cases} * & \text{if } x \in U \\ \phi & \text{if } x \notin U \end{cases}$

Let $g^x := g(\cdot)_x \circ y : \mathcal{O}(X) \longrightarrow \mathrm{Set}$
$U \longmapsto y(U)_x$.

$$\rightsquigarrow \quad \mathrm{Set}^{\mathcal{O}(X)^{op}} \underset{\mathrm{Lan}_y g^x}{\overset{R g^x}{\rightleftarrows}} \mathrm{Set} \; ; \; \mathrm{Lan}_y g^x \dashv R g_x .$$

<u>Notice</u> the $\overset{\text{stalk}}{\vee}$ functor $F \overset{(\cdot)_x}{\longmapsto} F_x = \varinjlim_{x \in U} F(U)$ is colimit preserving,

& $y(U) \longmapsto y(U)_x = g^x(U) \Rightarrow \mathrm{Lan}_y g^x \cong (\cdot)_x$

Also, $Rg^x(A)(U) \cong \mathrm{Hom}(y(U), Rg^x(A))$

$$\cong \mathrm{Hom}(\underset{g^x(U)}{\underset{\shortparallel}{g(U)_x}}, A) \cong \begin{cases} A & \text{if } x \in U \\ * & \text{if } x \notin U \end{cases}$$

$\Rightarrow Rg^x(A) \cong i \circ \mathrm{Sky}_x(A) \cong i \circ x_* A \Rightarrow Rg^x \cong i \circ x_*$, where $i : Sh(X) \hookrightarrow \mathrm{Set}^{\mathcal{O}(X)}$.

Some Categorical Remarks

Let $\mathcal{C}$ be a category, and $X \in \mathcal{C}_0$ an object. Consider the forgetful functor

$$p_X : \mathcal{C}/X \longrightarrow \mathcal{C}.$$

Suppose that $\forall\, Y \in \mathcal{C}$, $X \times Y$ exists.

Then p_X has a right adjoint, $Y \in \mathcal{C} \mapsto pr_1 : X \times Y \to X$.

$$\begin{tikzcd} Z \arrow[r, dashed, "h"] \arrow[dr, "f"'] & X \times Y \arrow[d] \arrow[dl, phantom, "\cong"] \\ & X \end{tikzcd} \quad \longleftrightarrow \quad Z \longrightarrow Y.$$

As a left adjoint, p_X preserves colimits, but more is true:

p_X <u>creates</u> colimits, i.e. p_X preserves <u>and reflects</u> colimits, and if $F : J \longrightarrow \mathcal{C}/X$, F has a colimit $\Longleftrightarrow$ $p_X F$ does.

<u>Sketch</u>: If $\rho : p_X F \Longrightarrow \Delta_Z$ is colimiting, then

$\tilde{F} : p_X F \Longrightarrow \Delta_X \quad \longleftrightarrow \quad Z \xrightarrow{\bar{F}} X \in \mathcal{C}/X$

$\tilde{F}(j) = F(j) : p_X F(j) \longrightarrow X$ and $\bar{F} \cong \varinjlim_J F$.

The forgetful functor

$$U : \mathbf{Top} \longrightarrow \mathbf{Set} \quad \text{creates colimits.}$$

Since both Top & Set are cocomplete, it suffices to look how colimits in Top are computed.

1st/ If $\{ f_\alpha : UZ_\alpha \longrightarrow X \}_\alpha$ is a collection of set-theoretic maps, $\rightsquigarrow$ a topology on X, called the <u>final topology</u> (wrt f_α's):

$$U \subset X \text{ is open} \Leftrightarrow f_\alpha^{-1}(U) \text{ is open in } Z_\alpha \ \forall \alpha.$$

It is the finest topology on X making each f_α continuous.

If $F : J \longrightarrow \mathbf{Top}$, compute $\varinjlim UF$ in Set.

It comes with a colimiting cocone

$$\rho : UF \Longrightarrow \Delta_{\varinjlim UF}$$

endow $\varinjlim UF$ with the final topology wrt the maps

$$\{ \rho(j) : UF(j) \longrightarrow \varinjlim UF \mid j \in J_0 \},$$

then $\varinjlim UF$ with this topology is $\varinjlim F$

(and each $\rho(j)$ becomes continuous $\rightsquigarrow \tilde{\rho} : F \Longrightarrow \Delta_{\varinjlim F}$ colimiting).

Since Set is cocomplete $\Rightarrow$ so is Top $\Rightarrow$ so is $\mathbf{Top}/X \ \forall X$.

The E'tale' space construction

Consider the functor

$$j : \mathcal{O}(X) \longrightarrow \mathrm{Top}/X$$
$$U \longmapsto U \hookrightarrow X \quad (\text{can. inclu.}).$$

$$\rightsquigarrow \quad \mathrm{Set}^{\mathcal{O}(X)^{op}} \underset{\Lambda := \mathrm{Lan}_y j}{\overset{R_j =: \Gamma}{\rightleftarrows}} \mathrm{Top}/X, \quad \Lambda \dashv \Gamma$$

<u>Notice</u>: $R_j(P \xrightarrow{\pi} X)(U) \cong \mathrm{Hom}_{\mathrm{Top}/X}(j(U), \pi)$

$$\cong \left\{ \begin{tikzcd} & P \arrow[d, "\pi"] \\ U \arrow[r, hook] \arrow[ur, dotted, "\sigma"] & X \end{tikzcd} \right\} = \Gamma(\pi)(U)$$

<u>Note</u>: If $\mathcal{D} \xrightarrow{G} \mathcal{E}$ is a colimit preserving functor between cocomplete categories, and $\underset{\text{is small}}{\mathcal{C}} \xrightarrow{f} \mathcal{D}$, by universal properties,

$$G \circ \mathrm{Lan}_y f \cong \mathrm{Lan}_y (Gf)$$ (both are colimiting pres. and agree on representables).

<u>Notice</u> The following diagram commutes:

$$\begin{tikzcd} \mathrm{Top}/X \arrow[r, "P_X"] \arrow[d, "U/X"'] & \mathrm{Top} \arrow[d, "U"] \\ \mathrm{Set}/UX \arrow[r, "P_{UX}"] & \mathrm{Set} \end{tikzcd}$$

all categories are cocomplete, and all functors preserve all colimits.

$\therefore$ if $\Lambda(F) = (E(F) \longrightarrow X)$, to compute $UE(F)$, it suffices to compute $\mathrm{Lan}_y(P_{UX} \circ U/X)(F)$.

<u>Note</u> $\quad \mathrm{Set}/UX \underset{\sim\ \psi}{\overset{\sim'}{\rightleftarrows}} \mathrm{Set}^{J^*UX} = \mathrm{Set}^{|UX|}$ (special case of 2 f) HW 1)

$$S \xrightarrow{f} UX \longmapsto (f^{-1}(y),\, y\in X)$$

$$\coprod_{y\in X} A_y \to X \longleftarrow\!\!\shortmid (A_y,\, y\in X)$$

and $p_{UX} \circ \psi(A_y, y\in X) = \coprod_{y\in X} A_y$.

In $\mathrm{Set}^{|UX|}$, colimits are computed pt-wise, so $\forall x\in X$, the functor $ev_x : \mathrm{Set}^{|UX|} \longrightarrow \mathrm{Set}$ is colimit preserving.

$$(A_y, y\in X) \longmapsto A_x$$

Lets put these together $\circ$:

$$\begin{tikzcd} \mathrm{Top}/X \arrow[r, "U/X"] \arrow[rrr, bend left, "=: fib_x"] & \mathrm{Set}/UX \arrow[r, "\psi", "\sim"'] & \mathrm{Set}^{|UX|} \arrow[r, "ev_x"] & \mathrm{Set} \end{tikzcd}$$

$$(P \xrightarrow{\pi} X) \longmapsto \pi^{-1}(x)$$

is colimit preserving.

$\Rightarrow\ \mathrm{Lan}_y (fib_x \circ j) \cong fib_x \circ \mathrm{Lan}_y j = fib_x \circ \Delta$

<u>Note</u>: $fib_x \circ j(U) = fib_x(U \hookrightarrow X) \cong \begin{cases} * & \text{if } x\in U \\ \phi & \text{if } x\notin U \end{cases} = y(U)_x$

$\Rightarrow\ fib_x \circ j \cong g^x \Rightarrow \mathrm{Lan}_y (fib_x \circ j) \cong (\cdot)_x$,

so $fib_x(\Delta(F)) \cong F_x$ – the stalk of F at x!

So, we know at the level of underlying sets,

$$\underline{\Delta(F)} = \coprod_{x\in X} F_x \xrightarrow{e_F} X.$$