Geometric Morphisms

Prop: Let $\mathbb{L}$ be a locale and let $1 \in Sh(\mathbb{L})$ be the terminal sheaf. Then there is an isomorphism of locales

$$\mathbb{L} \xrightarrow{\sim} Sub(1).$$

Pf Given $l \in \mathbb{L}$, the $l \xrightarrow{!} 1$ is mono since every arrow in a poset is, and $\mathbb{L} \underset{y}{\hookrightarrow} Set^{\mathbb{L}^{op}} \xrightarrow[a]{} Sh(\mathbb{L})$ is left exact $\Rightarrow$ $y(l) \longrightarrow 1$ is also mono.

$\rightsquigarrow$ $\underset{\text{(underlying sets)}}{\mathbb{L} \longrightarrow Sub(1)}$ which is clearly injective.

It also preserves arbitrary meets since y preserve all limits and any representable is a sheaf (we didn't show this, but it's easy).

Let $Sub(1) \longrightarrow \mathbb{L}$

$S \rightarrowtail 1 \longmapsto \bigvee\{l \in \mathbb{L} \mid S(l) \neq \emptyset\} =: V_S$

Claim $y(V_S) \cong S$. To see this, note that since S is a sheaf, if $(l_i)_i$ is a set s.t. $S(l_i) \neq \emptyset$ $\forall i \Rightarrow$ $S(\bigvee_i l_i) \neq \emptyset$, since if $S(l_i) \neq \emptyset \Rightarrow S(l_i) = 1$ and

$$S(\bigvee_i l_i) \xrightarrow{\sim} \lim(\prod_i S(l_i) \rightrightarrows \prod_{i,j} S(l_i \wedge l_j)) = \lim(* \rightrightarrows *) \cong *.$$

$\Rightarrow$ if $S(l_i) \neq \emptyset$ then $S(l_i \wedge l_j) \neq \emptyset$ and hence

$\therefore$ $S(V_S) \cong *$. (This argument shows $\mathbb{L} \longrightarrow Sub(1)$ preserves joins).

Now, if $S(U) \neq \emptyset \Rightarrow S(U) = *$ and $U \leq V_S \Rightarrow y(V_S)(U) = *$.

Conversely, if $y(V_S)(U) = * \Rightarrow U \leq V_S \Rightarrow \exists!\ \underset{\underset{*}{\text{\rotatebox{90}{$\cong$}}}}{S(V_S)} \longrightarrow S(U) \Rightarrow S(U) \neq \emptyset$

$\therefore$ $S(U) \cong *$. $\boxed{\xi}$

Let $f : \mathbb{L} \longrightarrow \mathbb{M}$ be a morphism of locales.

$\updownarrow$

$f^* : \mathbb{M} \longrightarrow \mathbb{L}$ in Frm.

Consider:

$$\begin{tikzcd}
\mathrm{Set}^{\mathbb{M}^{op}} \arrow[r, shift right, "\mathrm{Lan}_y(y\circ f^*)"'] & \mathrm{Set}^{\mathbb{L}^{op}} \arrow[l, shift right, "R_{y\circ f^*}"'] \\
\mathbb{M} \arrow[u, "y"] \arrow[r, "f^*"] & \mathbb{L} \arrow[u, "y"']
\end{tikzcd}$$

$$R_{y\circ f^*}(X)(m) \cong \mathrm{Hom}(y(m), R_{y\circ f^*}(X))$$

$$\cong X(f^*(m))$$

$$\Rightarrow R_{(y\circ f^*)}(X) = X\circ (f^*)^{op}.$$

<u>Prop</u> $R_{(y\circ f^*)}$ restricts to a functor

$$f_* : \mathrm{Sh}(\mathbb{L}) \longrightarrow \mathrm{Sh}(\mathbb{M}).$$

<u>Pf</u> f^* sends covers to covers.

<u>Prop</u> f_* has a left exact left adjoint.

<u>Pf</u> Consider the localizations

$$\begin{tikzcd}
\mathrm{Sh}(\mathbb{L}) \arrow[r, shift right, "i"'] & \mathrm{Set}^{\mathbb{L}^{op}} \arrow[l, shift right, "a"'] \\
\mathrm{Sh}(\mathbb{M}) \arrow[r, shift right, "j"'] & \mathrm{Set}^{\mathbb{M}^{op}} \arrow[l, shift right, "b"']
\end{tikzcd}$$

<u>Claim</u>: $a\circ \mathrm{Lan}_y(y\circ f^*)\circ j$ is left adjoint to f_*:

Let $F \in \mathrm{Sh}(\mathbb{L})$, $G \in \mathrm{Sh}(\mathbb{M})$.

$$\mathrm{Hom}_{\mathrm{Sh}(\mathbb{L})}(a\circ \mathrm{Lan}_y(y\circ f^*)\circ j(G), F) \cong \mathrm{Hom}_{\mathrm{Set}^{\mathbb{L}^{op}}}(\mathrm{Lan}_y(y\circ f^*)\circ j(G), iF)$$

$$\cong \mathrm{Hom}_{\mathrm{Set}^{\mathbb{M}^{op}}}(j(G), R_{(y\circ f^*)}\circ iF)$$

$$\cong \mathrm{Hom}_{\mathrm{Set}^{\mathbb{M}^{op}}}(j(G), j(f_* F)) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathbb{M})}(G, f_* G).$$

Note: a and j preserve all limits, so it suffices to show that $\mathrm{Lan}_y(y \circ f^{*})$ is left exact. But $\mathbb{M}$ has finite limits, $y \circ f^{*}$ is left exact and $\mathrm{Set}^{\mathbb{L}^{op}}$ is a topos (so satisfies Giraud's axioms) so the lemma from last time

$\Rightarrow \mathrm{Lan}_y(y \circ f^{*})$ is left exact.

Notation $\quad a \circ \mathrm{Lan}_y(y \circ f^{*}) \circ j =: f^*$

So we get that from a morphism of locales

$f : \mathbb{L} \longrightarrow \mathbb{M}$ a pair of adjoint functors

$$\mathrm{Sh}(\mathbb{L}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sh}(\mathbb{M}), \quad f^* \dashv f_*$$

s.t. f^* preserves finite limits.

Def Let $\mathcal{E}$ and $\mathcal{F}$ be Grothendieck topoi.

A geometric morphism $\mathcal{E} \xrightarrow{\varphi} \mathcal{F}$ consists of a pair of adjoint functors $\mathcal{E} \underset{\varphi_*}{\overset{\varphi^*}{\leftrightarrows}} \mathcal{F}$, $\varphi^* \dashv \varphi_*$ s.t. φ^* is left exact

Prop For $\mathbb{L}, \mathbb{M}$ locales, there is a natural bijection between

1) morphisms of locales $\mathbb{L} \longrightarrow \mathbb{M}$

2) isomorphism classes of geometric morphisms $\mathrm{Sh}(\mathbb{L}) \longrightarrow \mathrm{Sh}(\mathbb{M})$

Pf We have already seen that $f : \mathbb{L} \longrightarrow \mathbb{M}$ gives rise to a geometric morphism $\mathrm{Sh}(f) := (f_*, f^*) : \mathrm{Sh}(\mathbb{L}) \longrightarrow \mathrm{Sh}(\mathbb{M})$

Conversely, suppose that $(\varphi_*, \varphi^*): Sh(\mathbb{L}) \longrightarrow Sh(\mathbb{M})$

is a geometric morphism.

Both φ_* and φ^* are left exact $\Rightarrow$ they restrict to adjoint functors

$$\begin{tikzcd} Sub_{\mathbb{L}}(1) \arrow[r, shift left, leftarrow, "\overline{\varphi}^*"] \arrow[r, shift right, "\overline{\varphi}_*"'] & Sub_{\mathbb{M}}(1) \end{tikzcd}$$

with $\overline{\varphi}^*$ left exact since φ^* is. By the first proposition,

$$\Rightarrow \quad \mathbb{L} \xleftarrow{\overline{\varphi}^*} \mathbb{M} \text{ in } Frm.$$

It remains to show these constructions are mutually inverse.

Let $f: \mathbb{L} \to \mathbb{M}$. Since $f^*: \mathbb{M} \longrightarrow \mathbb{L}$ preserves arbitrary joins (colimits) $\exists!$ right adjoint $f_*: \mathbb{L} \longrightarrow \mathbb{M}$ (functor of posets).

It suffices to show f_* restricted to $Sub_{\mathbb{L}}(1)$ is f_*.

But for, $l \in \mathbb{L}$, $f_*(y(l)) = y(l) \circ (f^*)^{op}$, i.e.

$$f_*(y(l))(m) = Hom_{\mathbb{L}}(f^*(m), l) \cong Hom_{\mathbb{M}}(m, f_*(l))$$

$\Rightarrow f_*(y(l)) = y(f_*(l))$ as desired.

Conversely, let $\varphi = (\varphi_*, \varphi^*): Sh(\mathbb{L}) \longrightarrow Sh(\mathbb{M})$.

Let $\Psi = Sh(\overline{\varphi}^*)$. WTS $\Psi \cong \varphi$.

<u>Note</u>: For $F \in Sh(\mathbb{L})$,

$$\begin{aligned} \Psi_*(F)(m) &\cong F(\overline{\varphi}^*(m)) \cong Hom(y(\overline{\varphi}^*(m)), F) \\ &\cong Hom(\varphi^* y(m), F) \\ &\cong Hom(y(m), \varphi_* F) \cong (\varphi_* F)(m). \quad \boxed{\checkmark} \end{aligned}$$

Def Let $\widetilde{\mathcal{T}op}$ denote the following 2-category:

The objects are Grothendieck topoi. For two objects $\mathcal{E}, \mathcal{F}$,

$\mathrm{Hom}_{\widetilde{\mathcal{T}op}}(\mathcal{E}, \mathcal{F})$ is the following category:

<u>objects</u>: geometric morphism $(\varphi_*, \varphi^*) : \mathcal{E} \longrightarrow \mathcal{F}$

<u>arrows</u>: $\alpha : (\varphi_*, \varphi^*) \longrightarrow (\psi_*, \psi^*)$ are natural transformations $\alpha : \varphi^* \Longrightarrow \psi^*$.

Denote by $\mathcal{T}op'$ the (2,1)-category obtained by only considering natural <u>isomorphisms</u> above.

Rmk With a little care, one can improve the previous proposition to give an eq^l of categories

$$\mathrm{Hom}_{\widetilde{\mathbb{L}oc}}(\mathbb{L}, \mathbb{M}) \xrightarrow{\sim} \mathrm{Hom}_{\widetilde{\mathcal{T}op}}(Sh(\mathbb{L}), Sh(\mathbb{M})),$$

where $\nearrow$ has objects frame morphisms $\mathbb{M} \longrightarrow \mathbb{L}$ and arrows are natural transformations $\mathbb{M} \underset{}{\overset{}{\Downarrow}} \mathbb{L}$ (if they exist, they are unique).

In particular there is an eq^l

$$\underbrace{\mathrm{Hom}_{\mathrm{Loc}}(\mathbb{L}, \mathbb{M})}_{\text{set}} \xrightarrow{\sim} \underbrace{\mathrm{Hom}_{\mathcal{T}op'}(Sh(\mathbb{L}), Sh(\mathbb{M}))}_{\text{groupoid}}$$

so that $Sh : \mathrm{Loc} \hookrightarrow \mathcal{T}op'$ is full and faithful.

Def A Grothendieck topos is <u>localic</u> if it is in the essential image of Sh.

Rmk By composition, $\exists$ a functor

$$\mathrm{Top} \xrightarrow{\mathcal{O}} \mathrm{Loc} \xhookrightarrow{Sh} \mathfrak{Top}'$$

which is f&f when restricted to sober spaces.

Rmk For sober T_1 spaces (e.g. Hausdorff spaces) X, Y

$\mathrm{Hom}_{\widetilde{\mathbb{L}oc}}(\mathcal{O}(X), \mathcal{O}(Y))$ has only identity arrow, so

$$T_1\text{-Sober} \xhookrightarrow{Sh} \mathfrak{Top}.$$

(Non-identity morphisms in for non T_1-spaces are) useful in algebraic geometry.

Question: Given $\mathcal{E}$ a Groth. topos and $\mathbb{L}$ a locale, what do geometric morphism $\mathcal{E} \longrightarrow Sh(\mathbb{L})$ look like?

$\leftrightsquigarrow$ $f^*: Sh(\mathbb{L}) \longrightarrow \mathcal{E}$ colimit preserving and left exact,

so are determined by $f^* \circ \bar{y}: \mathbb{L} \longrightarrow \mathcal{E}$, $f^* = \mathrm{Lan}_{\bar{y}}(f^* \circ \bar{y})$

where $\bar{y} = (\mathbb{L} \xhookrightarrow[y]{} \mathrm{Set}^{\mathbb{L}^{op}} \xrightarrow{a} Sh(\mathbb{L}))$.

Note: $f^* \circ \bar{y}$ is left exact, so f^* is determined by a left exact functor $\Theta: \mathbb{L} \longrightarrow \mathcal{E}$.

When does such a left exact Θ induce a f^*?

$$\Theta \rightsquigarrow \begin{tikzcd} \mathrm{Set}^{\mathbb{L}^{op}} \arrow[r, shift right, "\mathrm{Lan}_y \Theta"'] & \mathcal{E} \arrow[l, shift right, "R_\Theta"'] \\ \mathbb{L} \arrow[u, hook, "y"] \arrow[ur, "\Theta"'] & \end{tikzcd}$$

claim:
which induces an adjunction

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "f_*"'] & Sh(\mathbb{L}) \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

$\Longleftrightarrow$ $R_\Theta(E)$ is a sheaf $\forall E \in \mathcal{E}$.

In such a situation, f^* will be left exact since $\mathrm{Lan}_y \Theta$ is.
since $\mathbb{L}$ and Θ are.

$\Leftarrow$ is clear and for $\Rightarrow$ given (φ_*, φ^*) consider

$$\begin{tikzcd} \mathrm{Set}^{\mathbb{L}^{op}} \arrow[r, shift right, "a"'] & \mathrm{Sh}(\mathbb{L}) \arrow[l, shift right, "i"'] \arrow[r, shift right, "\varphi^*"'] & \mathcal{E} \arrow[l, shift right, "\varphi_*"'] \\ \mathbb{L} \arrow[u, hook, "y"] \arrow[urr, "\Theta = \varphi^* \circ \bar{y}"'] & & \end{tikzcd}$$

$\varphi^* \circ a = \mathrm{Lan}_y \Theta$ by uniqueness. (both colimit preserving ...)

$R_\Theta = i \circ \varphi_*$ takes values in sheaves.

So, when is $R_\Theta(E)$ a sheaf $\forall E$?

$\Leftrightarrow$ $\forall$ $(l_\alpha \leq l)$ with $l = \bigvee_\alpha l_\alpha$,

$$R_\Theta(E)(l) \xrightarrow{\sim} \varprojlim \Big(\prod_\alpha R_\Theta(E)(l_\alpha) \rightrightarrows \prod_{\alpha,\beta} R_\Theta(E)(l_\alpha \wedge l_\beta) \Big) \quad \forall E$$

$$\mathrm{Hom}(\Theta(l), E) \cong R_\Theta(E)(l), \qquad \varprojlim \Big(\prod_\alpha \mathrm{Hom}(\Theta(l_\alpha), E) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Hom}(\Theta(l_\alpha \wedge l_\beta), E) \Big)$$

$$\cong \mathrm{Hom}\Big(\varinjlim \big(\coprod_{\alpha,\beta} \Theta(l_\alpha \wedge l_\beta) \rightrightarrows \coprod_\alpha \Theta(l_\alpha)\big), E\Big)$$

so $R_\Theta(E)$ is a sheaf $\forall E$ $\Leftrightarrow$ $\Big(\Theta(\bigvee_\alpha l_\alpha) = \varinjlim \big(\coprod_{\alpha,\beta} \Theta(l_\alpha \wedge l_\beta) \rightrightarrows \coprod_\alpha \Theta(l_\alpha)\big)$ $\forall$ l_α. $\star$

But note: Θ is left exact $\Rightarrow$ $\Theta(1) = 1$ and $\Theta(l \to 1) : \Theta(l) \to 1$ is mono,

so Θ factors through $\mathrm{Sub}_\mathcal{E}(1)$ $\rightsquigarrow$ $\tilde{\Theta} : \mathbb{L} \to \mathrm{Sub}_\mathcal{E}(1)$ left exact

(i.e. preserves finite meets) and $\star$ $\Rightarrow$ $\tilde{\Theta}$ preserves arbitrary joins

$\Rightarrow$ $\tilde{\Theta}$ is a map of frames $\rightsquigarrow$ $\hat{\Theta} : \mathrm{Sub}_\mathcal{E}(1) \to \mathbb{L}$ in Loc.

<u>Rmk</u> If $\mathcal{E} \simeq \mathrm{Sh}(\mathbb{M})$ and $\varphi = \mathrm{Sh}(f)$ for $f : \mathbb{M} \to \mathbb{L}$,

then $\mathrm{Sub}_\mathcal{E}(1) \cong \mathbb{M}$ and $\hat{\Theta} = f$.

<u>Cor</u>: $\mathrm{Sh} : \mathrm{Loc} \to \mathfrak{Top}$ has a left adjoint $\mathfrak{Top} \to \mathrm{Loc}$, $\mathcal{E} \mapsto \mathrm{Sub}_\mathcal{E}(1) =: \mathrm{Loc}(\mathcal{E})$

<u>Def</u> $\mathrm{Loc}(\mathcal{E})$ is called the <u>localic reflection</u> of $\mathcal{E}$.

Cor Set is the terminal object in $\mathcal{T}op$.

Pf Sh is a right adjoint so preserves limits.

If $\mathbb{F}$ is a frame, $\exists!\ \{0,1\} \xrightarrow{\varphi} \mathbb{F}$ since φ must preserve 0 and 1

$\Rightarrow \{0,1\} = \mathcal{O}(*)$ is terminal in Loc.

$\Rightarrow$ $Sh(\mathcal{O}(*)) \simeq Sh(*) \simeq Set$ is terminal in $\mathcal{T}op$. $\boxtimes$

Explicitly: If $\mathcal{E} \underset{\varphi_*}{\overset{\varphi^*}{\leftrightarrows}} Set$ is a geometric morphism, φ^* is left exact $\Rightarrow$ $\varphi^*(1) = 1$ and preserves colimits

$\Rightarrow$ $\varphi^*(X) \cong \varphi^*(\coprod_{x \in X} *) \cong \coprod_{x \in X} 1$.

Now, $\varphi_*(E) \cong Hom(1, \varphi_*(E)) \cong Hom(\varphi^* 1, E) \cong Hom(1, E)$.

Notation $\varphi^* = \Delta$, $\varphi_* = \Gamma$ = "global sections".

If $\mathcal{E} = Sh(Y)$, $\Delta(X)$ is the sheafification of the constant sheaf w/ value X, and $\Gamma(F) = F(Y)$ = global sections.

Def A geometric morphism $\mathcal{E} \xrightarrow{\varphi} \mathcal{F}$ is a geometric embedding if φ_* is full and faithful.

From the Homework (HW 7) we have the following corollary:

Cor There is a bijection between

1) (iso. classes of) geometric embeddings $\mathcal{E} \rightarrowtail Set^{\mathcal{C}^{op}}$

2) Grothendieck topologies on $\mathcal{C}$.