people.mpim-bonn.mpg.de/carchedi/topos.html

Motivation.

Intro

Q: What is a topos?

Answer: A topos is a generalized space.

The "problem" with schemes

$(X, \mathcal{O}_X)$ - locally ringed space, locally $\simeq \mathrm{Spec}_{Zar}(A)$

Zariski topology

Not enough open subsets for coho to capture much information.
Can't be fixed by a finer topology - the problem is with the notion of covering:

$(U_\alpha \hookrightarrow T)_\alpha$ ...

injections

Grothendieck: Find more general notion of space s.t. covers

$(\mathcal{E}_\alpha \to \mathcal{E})$ need not be inclusions.
All you need in order to define (sheaf!) cohomology is a notion of covering.

$\rightsquigarrow$ Grothendieck topoi.

Schemes $\rightsquigarrow$ Groth. top.

$X \longmapsto Sh(X_{et})$ $\leftsquigarrow$ coh. of this topos = étale coh.

local zeta functions of rational pts of varieties over $\overline{\mathbb{F}}_q$ gen. functions

(crucial step in proving the Weil conj.s)

(cx. var. $H^i_{sing}(X, \mathbb{Z}_\ell) \cong H^i_{et}(X, \mathbb{Z}_\ell)$ etc.)

and other fin. coeff.

analogue of RH $\rightsquigarrow$ Deligne

Not only do schemes have an underlying topos,
but also algebraic spaces, DM stacks ....

quotients of a scheme by certain group(oid) actions.

"pts" of a topos can have symmetries.

Also applications to differential topology:

$G \circlearrowright M \rightsquigarrow M /\!/ G$ is a topos (w/ a smooth atlas)

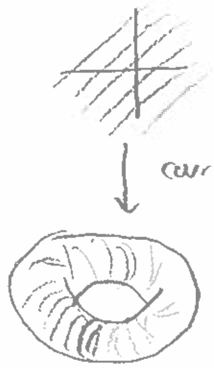
$\downarrow$ disc. gp

pts of a topos have symmetry groups, $\mathrm{Aut}[x] \cong G_x$ - stabilizer.

$\mathfrak{X}$ an orbifold is a topos "..."

Kronecker foliation

slope α



cov. pr.

each leaf dense & $\cong \mathbb{R}$

T

$T/\mathcal{F}$ indisc. $\rightsquigarrow$ no information.

$T/\!/\mathcal{F}$ - 1 dim'l smooth topos.

$T \longrightarrow T/\!/\mathcal{F}$ reconstructs foliation.

<u>Locales</u>: Spaces don't need an underlying set of points.

Most spaces are determined up to homeomorphism by their (abstract) lattice of open subsets

(counterexamples: an inf. set w/ cof. topology, indisc. space, some varieties)

Locale = a "topology" not tied to a set of points:

Doesn't need to <u>have</u> pts

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}/\mathbb{Q} \arrow[d] \\ \mathbb{Q} \arrow[r, hook] & \mathbb{R} \end{tikzcd}$$

p.b. in locales.

P: largest subspace of $\mathbb{Q}$ with no pts.

<u>Every</u> locale is a subspace of a top'l space.

P has no pts, but has an interesting topology.

$$\begin{tikzcd} & \mathcal{T}op_0 \arrow[d, phantom, "\shortparallel"] & & & & \\ \mathrm{Top} \arrow[r, shift left] & \mathrm{Loc} \arrow[l, shift left] \arrow[r, hook, shift right] & \mathcal{T}op_1 \arrow[l, shift right] \arrow[r, hook, shift right] & \mathcal{T}op_2 \arrow[l, shift right] \arrow[r, hook, shift right] & \mathcal{T}op_3 \arrow[l, shift right] \cdots & \mathcal{T}op_\infty \end{tikzcd}$$

$\uparrow$ ff. on all "reasonable" spaces

$\uparrow$ Stop here

Every topos is a ~~quotient~~ of a locale by a groupoid action.

<u>Answer</u>: A topos is a category that behaves like the category Set.

E.g.

0) Set

1) G-Set, $G \circlearrowright X$ G gp, $*//G$ for trivial G-action.

1') M-Set, M a monoid

2) Sh(X), X-space "continuously varying sets /X" "=" X when viewed as a topos.

3) sequential sets $X_0 \to X_1 \to X_2 \to X_3 \to \cdots$

4) simplicial sets $X_0 \rightleftarrows X_1 \rightrightarrows X_2 \cdots$

5) category of graphs

6) Set / X (Zermelo-Fraenkel)

You can perform "set theoretic" constructions in all of these categories

limits & colimits "behave like in Set"

can talk about subsets, power sets, the "set" of functions

even have an object of "natural numbers" $\mathbb{N}$ & "real numbers" $\mathbb{R}$

<u>Answer</u>: <u>A topos is a category of spaces</u>

Sh(Mfd) ⇝ $\boxed{\text{contains}}$ all inf. dim'l mfds

Banach Mfd, Fréchet Mfd, Diffeological spaces, Frölicher spaces

– also $\forall X, Y$ (possibly ∞-dim'l) contains $\mathrm{Map}(X,Y)$ – a "smooth space".

$\mathcal{D}$ = Dubuc topos $\supset$ <u>infinitesimal</u> manifolds

e.g. $\exists\ D = \{\bullet\!\!\to\}$ (infinitesimal vector), $\forall M$ mfd, $M^D = \{f : D \to M\}$ is a manifold in $\mathcal{D}$

<u>In fact</u>, $M^D = TM$.

Answer: Topoi classify algebraic structures.

$\exists$ a topos $\mathcal{E}$ s.t. a morphism $\mathcal{F} \longrightarrow \mathcal{E}$ is the same as an abelian group in $\mathcal{F}$ (ab. gp object.)

" " " ring " "

" " local ring " "

" " monoid " "

" category " "

$\mathcal{E}$ = classifying topos of

In fact, $\mathcal{T}op$ is a 2-cat. and $Hom(\mathcal{F}, \mathcal{E}) \simeq$ Category of objects in $\mathcal{F}$.

e.g. $\mathcal{E}$ = classifying topos for rings

$\mathcal{E}$ carries the "universal ring object" U and

every ring object $R \in \mathcal{F} \rightsquigarrow f: \mathcal{F} \longrightarrow \mathcal{E}$ s.t. $f^*U \cong R$.

Note A map $X \to \mathcal{E} \rightsquigarrow$ a sheaf of rings on X (since X is really $Sh(X)$)

$\mathcal{E} = G\text{-}Set$ = classifying topos for principal G-bundles.

U = universal principal G-bundle.

X a space (or topos) $Hom(X, \mathcal{E}) \simeq Bun_G(X)$ as categories (*)

no homotopy classes here!

Moreover $\exists$ a map $BG \longrightarrow \mathcal{E}$ is a w.h.e. (**)

classifying space

Finally $\exists$ a topos Γ s.t. a "local homeomorphism" $X \longrightarrow \Gamma$ with X a space $\rightsquigarrow$ a smooth atlas on X.